TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 263, Number 2, February 1981

PRODUCT-CONVOLUTION OPERATORS AND MIXED-NORM SPACES

BY

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ABSTRACT. Conditions for boundedness and compactness of product-convolution operators $g \to P_h C_f g = h \cdot (f * g)$ on spaces $L_p(G)$ are studied. It is necessary for boundedness to define a class of "mixed-norm" spaces $L_{(p,q)}(G)$ interpolating the $L_p(G)$ spaces in a natural way ($L_{(p,p)} = L_p$). It is then natural to study the operators acting between $L_{(p,q)}(G)$ spaces, where G has a compact invariant neighborhood. The theory of $L_{(p,q)}(G)$ is developed and boundedness and compactness conditions of a nonclassical type are obtained. It is demonstrated that the results extend easily to a somewhat broader class of integral operators. Several known results are strengthened or extended as incidental consequences of the investigation.

1. Introduction. Convolution by f in $L_1(R)$ defines a bounded operator, C_f, on $L_p(R)$ for $1 \leqslant p \leqslant \infty$; likewise, pointwise multiplication by h in $L_\infty(R)$ defines a bounded operator P_h on $L_p(R)$. Excepting trivial cases, these operators are never compact. The composition $P_h C_f$ of two such operators, which we term a product-convolution (PC) operator, is frequently compact. Asking exactly when this occurs motivates this paper. PC operators on L_p of a locally compact group arise in many areas of analysis. In [2] and [3] the compactness of certain PC operators was used to study induced representations of locally compact groups. PC operators (and their adjoint CP operators) arise naturally in many applied problems.

Even with h in $L_\infty(R)$ and f in $L_1(R)$, we find that the "mixed-norm" spaces of [1] must be introduced to solve the compactness problem for $P_h C_f$. These spaces also arise unavoidably when one attempts to weaken conditions on h and f and keep $P_h C_f$ bounded. Various mixed-norm conditons on h and f guarantee boundedness of $P_h C_f$ from L_p to L_q and, in fact, between mixed-norm spaces. Conversely, $P_h C_f$ may be bounded from L_p to L_q (or from one mixed-norm space to another) with neither h nor f in any L_r, but both h and f must belong to mixed-norm spaces. Thus mixed-norm spaces are the natural setting for studying bounded PC operators. We treat a wide class of PC operators and, in all but one case, find necessary and sufficient conditions for compactness. The conditions involve only membership in a mixed-norm space or, in one instance, translational continuity in mixed-norm spaces; they are usually easily checked for explicit examples. By applying various manipulations to PC operators, we develop easily applied necessary and sufficient conditions for compactness of a wider variety of integral operators which

Received by the editors January 10, 1978 and, in revised form, January 11, 1980.
AMS (MOS) subject classifications (1970). Primary 43A15, 47B05; Secondary 22D15, 43A60.

do not fall within the scope of classical compactness theorems. We attempt reasonable generality but do not pursue the extensions to vector-valued functions.

§2 establishes notation and reviews needed facts. In §3 we define "uniform partitions" on locally compact groups and show they exist. We then use these partitions to define and study certain mixed-norm spaces. Many of these spaces on the real line have been studied by Holland **[8]**, **[9]**. Kellogg **[10]** and Williams **[17]** studied analogous spaces on certain discrete groups. The paper **[18]** of Bertrandias, Datry, and Depuis appeared while this paper was in a referee's hands. It independently develops, for abelian groups G, spaces $l^q(L^p)$ equivalent to our mixed-norm spaces $L_{(p,q)}(G)$. We hoped in revision to use their definition, which cleverly avoids uniform partitions, but we found this obscured the mixed-norm structure and complicated our computations. Our techniques for nonabelian G require the uniform partition and mixed-norm structure as do some spaces we use which are not equivalent to those of **[18]** even for abelian G. We have therefore not adopted the elegant approach of **[18]**, which was developed for different ends.

The general construction of mixed-norm spaces as function spaces on products was given by Benedek and Panzone **[1]**. As a Borel space, R is the product of an interval and the integers. The space $\underline{L}^{(p,q)}(I \times Z)$ of **[1]** consists of functions which are in $L_p([n, n + 1])$ for each n and such that the resulting sequence of $L_p(I)$ norms is in l_q. Emerson and Greenleaf **[5]** show every locally compact group to be the union of disjoint translates of some relatively compact Borel neighborhood. If the group is second countable it is borel isomorphic to the product of a discrete set with this neighborhood, so the construction on R generalizes. Unfortunately this partition may not be conveniently related to group properties; moreover, we need arbitrarily fine partitions and it is not known that arbitrarily fine "Emerson-Greenleaf neighborhoods" exist. These problems, rather than hesitancy to assume second countability, lead us to take an alternative approach in §3.

Our $L_{(p,\infty)}$ seems a natural place to study almost periodic functions and stationary stochastic processes on G. (The Stepanovich a.p. functions are the closures in $L_{(p,\infty)}(R)$ of the trigonometric polynomials.) Particular mixed-norm spaces on R^n studied by N. Wiener, R. Goldberg, and P. Szeptycki are cited in **[18]**.

§4 investigates convolution of functions from mixed-norm spaces and generalizes Young's inequality. In arbitrary locally compact groups, mixed-norm properties may be obscured by convolution. To prevent this, we assume the group is (IN), i.e., has a compact invariant neighborhood of the identity. This classical condition **[6]** is equivalent to existence of a nontrivial center in the group algebra **[13]**. (It seems unlikely (IN) is the weakest useful condition, but attempting to use unimodularity alone proved intractable.)

Our general Young's inequality shows the $L_{(p,1)}(G)$ to be closed under convolution and hence a group algebra. It enables us to strengthen a result of Stewart **[16]** to show that there is always a bounded L_1 function on an abelian group having its Fourier transform of compact support and identically equal to one on a prescribed compact set. We also improve a result of Rickert **[14]**, who showed that on a locally compact, noncompact group if $1/p + 1/q < 1$ there are f in L_p and g in L_q such that $f * g$ fails to exist on a set of positive measure. We give precise conditions on f for this and show the set of such f to be of second category in L_p.

In §5 we consider the PC operators which are bounded by amalgams of classical estimates and we show that if $P_h C_f$ is bounded h and f must belong to mixed-norm spaces.

In §6 we give necessary and sufficient conditions for compactness of all the operators studied in §5 with the exception of the case where the range space has infinite indices. (The difficulty there is lack of useable compactness criteria in L_∞.) The conditions for compactness of $P_h C_f$ are mostly simply membership of h and f in ordinary mixed-norm spaces $L_{p,q}(G)$, but in one exceptional case a different space involving a "local-multiplier" condition must be used. The section concludes with a number of examples. §7 summarizes our results with particular application to $L_2(R)$ and demonstrates the application to a wider class of operators.

2. Notation and preliminary remarks. We will always discuss a locally compact group, G. Functions are complex-valued and measurable unless otherwise specified. Measurability and integration is relative to a given left Haar measure m. If $1 \leqslant p \leqslant \infty$ then p' will denote the conjugate index: $1/p + 1/p' = 1$. We denote the continuous complex-valued functions on G by $C(G)$, while $C_\infty(G)$ and $K(G)$ denote functions vanishing at infinity and functions of compact support, respectively. For functions f on G we adopt the notations: $f^\sim(x) = f(x^{-1})$, $({}_yf)(x) = f(yx)$, and $f_y(x) = f(xy)$. By f_E we always mean the product of $f\chi_E$ of f by the characteristic function of a set E. If E is Borel and of positive measure, there is automatic identification of f_E with its restriction to E which imbeds $L_p(E)$ (for the restricted Haar measure) in $L_p(G)$ and we make the identification routinely.

We will need pairs of extended real indices which we denote $\underline{v} = (v_1, v_2)$. Inequalities such as $\underline{v} < \underline{u}$ are interpreted termwise and, if g is a function of one variable, $g(\underline{v})$ means $(g(v_1), g(v_2))$. If r is a specific extended real number, we will denote the pair (r, r) by $\bar{r}$. For any pair $\underline{v}$, we denote by $\underline{v}'$ the conjugate pair defined by $1/\underline{v} + 1/\underline{v}' = \bar{1}$. We will use a partial ordering $\succ$ defined by $(a_1, b_1) \succ (a_2, b_2)$ iff $a_1 \leqslant a_2$ and $b_1 \geqslant b_2$. For any set S of functions, we denote the nonnegative members by S^+.

We use the following repeatedly.

PROPOSITION 2.1. *Let* $\{a_1, a_2, \ldots, a_n\}$ *be nonnegative.*

(a) *If* $1 \leqslant p < \infty$ *and* $1/p + 1/q = 1$, *then*

$$n^{-1/q}(a_1 + a_2 + \cdots + a_n) \leqslant (a_1^p + a_2^p + \cdots + a_n^p)^{1/p} \leqslant (a_1 + a_2 + \cdots + a_n).$$

(b) *If* $0 < p \leqslant 1$ *and* $1/p + 1/q = 1$, *then*

$$(a_1 + a_2 + \cdots + a_n) \leqslant (a_1^p + a_2^p + \cdots + a_n^p)^{1/p} \leqslant n^{-1/q}(a_1 + a_2 + \cdots + a_n).$$

PROOF. (a) Apply Hölder's inequality and the monotonicity of norms to the measure space formed by assigning unit measure to the n integers.

(b) Apply (a) to $1/p$ and the numbers a_i^p.

3. Mixed norm spaces. To separate local and global properties of functions, we partition G into subsets not too dissimilar in "size and shape".

DEFINITION 3.1. Let U, V be relatively compact open neighborhoods of the identity with $\bar{U} \subset V$. A partition π of G into disjoint Borel subsets is U-V uniform if for each E in π there is x such that $xU \subset E \subset xV$. A Borel partition is uniform if it is U-V uniform for some (U, V).

LEMMA 3.2. *Let F be a subset of G such that for some fixed relatively compact open neighborhood U of the identity xU and yU are disjoint whenever x and y are distinct members of F. For each pair (K, L) of relatively compact Borel sets there is an integer $n_F(K, L) \geqslant 1$ such that each left translate of L intersects at most $n_F(K, L)$ of the sets xK for x in F.*

PROOF. Let $n_F(K, L)$ be the greatest integer in the real number $m(LK^{-1}U)/m(U)$. If $xK \cap x'L \neq \varnothing$ then $x \in x'LK^{-1}$ so $xU \subset x'LK^{-1}U$. Thus, if x_iK intersect $x'L$, for x_i in F ($i = 1, 2, \ldots, n$), the disjoint sets x_iU are in $x'LK^{-1}$, so $nm(U) \leqslant m(x'LK^{-1}U) = m(LK^{-1}U)$. Hence $n \leqslant n_F(K, L)$.

We now show that uniform partitions exist in abundance.

PROPOSITION 3.3. *Let U be a symmetric, open, relatively compact neighborhood of the identity. There exists a U-U^2 uniform partition of G.*

PROOF. Let $\mathfrak{T}$ be a maximal family of pairwise disjoint left translates of U, $\mathfrak{T} = \{x_iU_i\colon i \in I\}$ where I is well-ordered with first element i_0. For any x, $xU \cap x_iU \neq \varnothing$ for some i, so $G = \bigcup_{i\in I} x_iU^2$. Let $W_{i_0} = (x_{i_0}U^2) \sim (\bigcup_{i\neq i_0} x_iU)$ (where $\sim$ denotes set difference). Since $x_{i_0}U^2$ meets only finitely many x_iU by Lemma 3.2, W_{i_0} is Borel. Suppose a Borel set W_i has been defined for each $i < j$ such that

(α_i) $\bigcup_{k\leqslant i} W_k = (\bigcup_{k\leqslant i} x_kU^2) \sim (\bigcup_{k>i} x_kU)$.

(β_i) $\{W_k | k \leqslant i\}$ are pairwise disjoint.

(γ_i) $W_k \subseteq x_kU^2$ for $k \leqslant i$.

We then define $W_j = (x_jU^2) \sim [(\bigcup_{i<j} W_i) \cup (\bigcup_{k>j} x_kU)]$. By Lemma 3.2, x_jU^2 meets only finitely many of the W_i and the x_kU, so W_j is Borel. Thus W_i is defined by transfinite induction for all i so that (α_i), (β_i) and (γ_i) hold, but $\bigcup_{i\in I} W_i = \bigcup_{i\in I} x_iU^2 = G$ and $x_iU \subset W_i \subset x_iU^2$, so $\{W_i\}$ is a U-U^2 uniform partition.

Note that any U-V uniform partition π provides a situation in which Lemma 3.2 applies, since for each E in π we have $x_EU \subset E \subset x_EV$ and F can be taken as one set of such x_E. The following results are immediate.

PROPOSITION 3.4. *Let π and F be as above, K relatively compact and Borel. Every translate of K intersects (and so is covered by) at most $n_F(V, K)$ members of π.*

PROPOSITION 3.5. *Under the above conditions each left translate of a member of π intersects at most $n_F(K, V)$ of the translates x_EK.*

DEFINITION 3.6. For any $\underline{p} = (p_1, p_2)$, $\bar{1} \leqslant \underline{p} \leqslant \overline{\infty}$, and uniform partition π, we say f belongs to $L^\pi_{\underline{p}}(G)$ if $\bar{f}_E \in L_{p_1}(E)$ for all E in π and the family of norms $\{\|f_E\|_{p_1}\colon E \in \pi\}$ belongs to $l_{p_2}(\pi)$. (We say f is locally L_{p_1} and globally l_{p_2} relative to

π. The G is usually supressed and we write only $L^\pi_{\underline{p}}$.) We define the norm $\|f\|^\pi_{\underline{p}}$ to be the l_{p_2} norm of $\{\|f_E\|_{p_1}: E \in \pi\}$. (It is easily seen that $L^\pi_{\underline{p}}$ is a Banach space.)

We denote by $L^\pi_{(p_1,0)}$ the closed subspace of $L^\pi_{(p_1,\infty)}$ consisting of functions locally in L_{p_1} such that $\{\|f_E\|_{p_1}: E \in \pi\}$ vanishes at ∞ on π.

PROPOSITION 3.7 (CF. [**18**, §7]). (a) *For any π, and $1 \leqslant p \leqslant \infty$, $L^\pi_{(p,p)}$ is identical (and isometric) with $L_p(G)$.*

(b) *If $\underline{1} \leqslant \underline{p} \leqslant \overline{\infty}$, $f \in L^\pi_{\underline{p}}$, $g \in L^\pi_{\underline{p}'}$, then $fg \in L_1(G)$ and $\|fg\|_1 < \|f\|^\pi_{\underline{p}}\|g\|^\pi_{\underline{p}'}$. The mapping F_g: $f \to \int_G |fg|\, dm$ is a bounded linear functional, and if $\underline{p} < \overline{\infty}$ the correspondence $g \to F_g$ is an isometric isomorphism of $L^\pi_{\underline{p}'}$ onto the dual of $L^\pi_{\underline{p}}$. Similarly, the dual of $L^\pi_{(p,0)}$ can be canonically identified with $L^\pi_{(p',1)}$.*

(c) *If $\underline{p} \prec \underline{q}$, then $L^\pi_{\underline{p}} \subset L^\pi_{\underline{q}}$.*

(d) *If $1 \leqslant p \leqslant q \leqslant \infty$, then $L^\pi_{(p,q)} \supset L_p(G) \cup L_q(G)$ and $L^\pi_{(q,p)} \subset L_p(G) \cap L_q(G)$.*

(e) *If $\underline{p} < \overline{\infty}$, $K(G)$ is dense in $L^\pi_{\underline{p}}$. Moreover, if $p_1 < \infty$, $L^\pi_{(p_1,0)}$ is the closure of $K(G)$ in $L^\pi_{(p_1,\infty)}$.*

PROOF. (a) is immediate from the definition, while (c) and (d) follow from the definition and elementary norm inequalities. Because the norms depend on π, there is some interest to the proof of (b). Applying Minkowski's inequality first on each E in π and then on the discrete set π, we have

$$\int_G |fg|\, dm = \sum_{E \in \pi} \int_E |f_E g_E|\, dm \leqslant \sum_{E \in \pi} \|f_E\|_{p_1}\|g_E\|_{p_1'} \leqslant \|f\|^\pi_{\underline{p}}\|g\|^\pi_{\underline{p}'}$$

which proves the first part of (b) and shows F_g is in the dual of $L^\pi_{\underline{p}}$.

If ψ is in the dual of $L^\pi_{\underline{p}}$ it restricts to a bounded linear functional ψ_E on $L_{p_1}(E)$ for each E in π. By the classical result, for some g_E in $L_{p_1}(E)$, $\|g_E\|_{p_1'} = \|\psi_E\|$ and $\psi_E(f) = \int_E f g_E\, dm$ for all f in $L_{p_1}(E)$. Let g be the measurable function defined by setting $g(x) = g_E(x)$ for x in E and note that for f in $L^\pi_{\underline{p}'}$, $\psi(f) = \int_G fg\, dm$. We need only show $\|g\|^\pi_{\underline{p}'} = \|\psi\|$. Clearly $\|\psi\| \leqslant \|g\|^\pi_{\underline{p}'} < \infty$ by the first part of the proof. Given $\varepsilon > 0$, choose, for each E, k_E in $L_{p_1}(E)$ such that $\|k_E\|_{p_1} = 1$ and $\psi_E(k_E) = |\psi_E(k_E)| > (1 - \varepsilon)\|g_E\|_{p_1'}$. For any function $\{\beta_E\}$ in l_{p_2} of π, define $f(x) = |\beta_E| k_E(x)$ for x in E, a member of $L^\pi_{\underline{p}}$. Then

$$\|\psi\|\ \|\beta\|_{p_2} = \|\psi\|\ \|f\|^\pi_{\underline{p}} \geqslant |\psi(f)| = \left| \sum_{E \in \pi} |\beta_E|\ |\psi_E(k_E)| \right|$$

$$= \sum_{E \in \pi} |\beta_E|\ |\psi_E(k_E)| \geqslant (1 - \varepsilon) \sum_{E \in \pi} |\beta_E|\ \|g_E\|_{p_1'}.$$

Since β is arbitrary in $l_{p_2}(\pi)$ it follows that $(1 - \varepsilon)\|g_E\|_{p_1'}$ is in $l_{p_2'}(\pi)$ and has norm not greater than $\|\psi\|$. Since ε is arbitrarily small, however, $\|g\|^\pi_{\underline{p}'} \leqslant \|\psi\|$. The proof for $L^\pi_{(p_1,0)}$ is similar.

(e) For any f in $L^\pi_{\underline{p}}$ there is a finite $\alpha \subset \pi$ such that

$$\left[\sum_{E \notin \alpha} (\|f_E\|_{p_1})^{p_2} \right]^{1/p_2} < \varepsilon/2$$

so, writing $F = \bigcup_{E \in \alpha} E$, $\|f - f_F\|^\pi_{\underline{p}} < \varepsilon/2$.

On the finite set α the l_{p_1} and l_{p_2} norms are equivalent since they are norms on the same finite-dimensional vector space, so on F, $\|\cdot\|^{\alpha}_{(p_1,p_2)}$ is equivalent to $\|\cdot\|^{\alpha}_{(p_1,p_1)} = \|\cdot\|_{p_1}$. Since $K(G)$ is dense in $L_{p_1}(G)$ it is also dense in $L_{p_1}(F)$ and we can find g in $K(G)$ such that $\|f_F - g_F\|_{p_1}$ and hence

$$\|f_F - g_F\|^{\alpha}_{(p_1,p_2)} = \|f_F - g_F\|^{\pi}_{(p_1,p_2)}$$

is arbitrarily small.

Without loss of generality g vanishes outside an open $H \supset F$ with $m(H \sim F)$ arbitrarily small. Recall we can find a closed $K \subset F$ and an open $H \supset F$ such that $m(H \sim K)$ is arbitrarily small and a continuous h, $h = 1$ on K, $h = 0$ on $G \sim H$, $0 \leqslant h \leqslant 1$. With g above so that $\|f_F - g_F\|_{p_1} < \delta$, hg is in $K(G)$ and

$$\|f_F - (hg)_F\|_{p_1} \leqslant \|f_F - g_F\|_{p_1} + \|g_F - (hg)_F\|_{p_1} < \delta + \max|g|(m(F - K))^{1/p_1}.$$

By choice of K this is arbitrarily close to δ and g can be replaced by hg. For $i = 1, 2$, $\|g - g_F\|_{p_i} \leqslant \max|g|(m(H - F))^{1/p_i}$ can be made arbitrarily small by choice of H. If $p_2 \geqslant p_1$, by the norm inequality for discrete spaces,

$$\|g - g_F\|^{\pi}_{(p_1,p_2)} \leqslant \|g - g_F\|^{\pi}_{(p_1,p_1)} = \|g - g_F\|_{p_1}.$$

If $p_1 \leqslant p_2$, the reverse norm inequality, which holds for the finite measure spaces $E \in \pi$, yields (for some C)

$$\|g - g_F\|^{\pi}_{(p_1,p_2)} \leqslant C\|g - g_F\|^{\pi}_{(p_2,p_2)} = C\|g - g_F\|_{p_2}.$$

Thus we can choose g in $K(G)$ so that $\|f_F - g_F\|^{\pi}_{(p_1,p_2)} < \varepsilon/4$ and $\|g - g_F\|^{\pi}_{(p_1,p_2)} < \varepsilon/4$ so $\|f_F - g\|^{\pi}_{(p_1,p_2)} < \varepsilon/2$. Then $\|f - g\|^{\pi}_{(p_1,p_2)} \leqslant \|f - f_F\|^{\pi}_{(p_1,p_2)} + \|f_F - g\|^{\pi}_{(p_1,p_2)} < \varepsilon$. Thus $K(G)$ is dense in $L^{\pi}_{(p_1,p_2)}(G)$. The proof that it is dense in $L^{\pi}_{(p_1,0)}(G)$ is similar.

We now show the norms $\|\cdot\|^{\pi}_{\underline{p}}$ are equivalent as π varies, so the topological vector spaces $L^{\pi}_{\underline{p}}(G)$ are intrinsic to G.

PROPOSITION 3.8. *For π', π uniform partitions, $\bar{1} \leqslant \underline{p} \leqslant \overline{\infty}$, there is M such that $\|f\|^{\pi'}_{\underline{p}} \leqslant M\|f\|^{\pi}_{\underline{p}}$ for all measurable f.*

PROOF. For $E' \in \pi'$, let $S(E') \subset \pi$ be the set of $E \in \pi$ such that $E \cap E' \neq \varnothing$. For $E \in \pi$, let $T(E) \subset \pi'$ be the set of E' such that $E \in S(E')$. By 3.2 and corollaries, the cardinals of $S(E')$ and $T(E)$ have a common bound, M. Since $\|f_{E'}\|_{p_1} \leqslant \sum_{E \in S(E')}\|f_E\|_{p_1}$ for all E' in π', it is immediate that $\|f\|^{\pi'}_{\underline{p}} \leqslant M\|f\|^{\pi}_{\underline{p}}$ if $p_2 = \infty$. For $p_2 < \infty$, 2.1 yields

$$(\|f_{E'}\|_{p_1})^{p_2} \leqslant \left(\sum_{E \in S(E')} \|f_E\|_{p_1}\right)^{p_2} \leqslant M^{p_2 - 1} \sum_{E \in S(E')} (\|f_E\|_{p_1})^{p_2}$$

so

$$\begin{aligned}\|f\|^{\pi'}_{\underline{p}} &\leqslant M^{(p_2-1)/p_2}\left(\sum_{E' \in \pi'} \sum_{E \in S(E')} (\|f_E\|_{p_1})^{p_2}\right)^{1/p_2} \\ &= M^{(p_2-1)/p_2}\left(\sum_{E' \in T(E)} \sum_{E \in \pi} (\|f_E\|_{p_1})^{p_2}\right)^{1/p_2} \\ &\leqslant M\left(\sum_{E \in \pi} (\|f_E\|_{p_1})^{p_2}\right)^{1/p_2} = M\|f\|^{\pi}_{\underline{p}}.\end{aligned}$$

Next we give an intrinsic characterization of the spaces $L_{\underline{p}}^{\pi}$ and incidentally introduce new equivalent norms: $\|(|f|^{p_1} * g)^{1/p_1}\|_{p_2}$.

PROPOSITION 3.9. *The following are pairwise equivalent for* $1 \leq p_1 < \infty$:

(a) *f is in* $L_{\underline{p}}^{\pi}$.

(b) *For some $g \not\equiv 0$ in $K^+(G)$, $(|f|^{p_1} * g)(x)$ exists for all x in G and $(|f|^{p_1} * g)^{1/p_1}$ is in $L_{p_2}(G)$.*

(c) *For all g in $K^+(G)$, $(|f|^{p_1} * g)(x)$ exists for all x in G and $(|f|^{p_1} * g)^{1/p_1}$ is in $L_{p_2}(G)$.*

PROOF. Clearly (c) implies (b). We show (a) implies (c). Let π and F be as in 3.4. Suppose $f \in L_{\underline{p}}^{\pi}$, $g \in K^+(G)$, K the support of g and $\beta = \max_x |g(x)|$. Let $T(E) = \{E' \in \pi: E' \cap x_E VK^{-1} \neq \varnothing\}$ have cardinality $C_1(E)$ and $S(E) = \{E' \in \pi: E \in T(E')\}$ have cardinality $C_2(E)$. If $x \in E \in \pi$, then

$$(|f|^{p_1} * g)(x) = \int_G |f(y)|^{p_1} g(y^{-1}x)\, dm(y) \leq \beta \int_{x_E VK^{-1}} |f(y)|^{p_1}\, dm(y)$$
$$\leq \beta \sum_{E' \in T(E)} (\|f_{E'}\|_{p_1})^{p_1} < \infty.$$

By 3.2, $C_1(E) \leq n_F(V, VK^{-1})$ so, for $1 \leq p_2 < \infty$, 2.1 yields

$$\left[(|f|^{p_1} * g)(x)\right]^{1/p_1} \leq \beta^{1/p_1} \sum_{E' \in T(E)} \|f_{E'}\|_{p_1}$$
$$\leq \left(n_F(V, VK^{-1})\right)^{(p_2-1)/p_2} \beta^{1/p_1} \left[\sum_{E' \in T(E)} (\|f_{E'}\|_{p_1})^{p_2}\right]^{1/p_2}.$$

Then

$$\int_G \left[(|f|^{p_1} * g)(x)\right]^{p_2/p_1} dm(x) = \sum_{E \in \pi} \int_E \left[(|f|^{p_1} * g)(x)\right]^{p_2/p_1} dm(x)$$
$$\leq m(V)\left(n_F(V, VK^{-1})\right)^{(p_2-1)} \beta^{p_2/p_1} \sum_{E \in \pi} \sum_{E' \in T(E)} (\|f_{E'}\|_{p_1})^{p_2}$$
$$= m(V)\left(n_F(V, VK^{-1})\right)^{(p_2-1)} \beta^{p_2/p_1} \sum_{E \in S(E')} \sum_{E' \in \pi} (\|f_{E'}\|_{p_1})^{p_2}$$
$$\leq m(V)\left(n_F(V, VK^{-1})\right)^{(p_2-1)} \beta^{p_2/p_1} n_F(VK^{-1}, V)(\|f\|_{\underline{p}}^{\pi})^{p_2} < \infty$$

since $C_2(E') \leq n_F(VK^{-1}, V)$ by 3.5. Thus (a) implies (c). (The proof for $p_2 = \infty$ is similar.)

Next we show (b) implies (a). (We show the more difficult case $p_2 < \infty$.) Let π and F be as above, and let $g \not\equiv 0$ in $K^+(G)$ be such that $(|f|^{p_1} * g)^{1/p_1}$ exists everywhere and lies in $L_{p_2}(G)$.

For some $\delta > 0$, the open, relatively compact set $W = \{x: g(x) > \delta\} \neq \varnothing$, so, for all x, $f_{xW^{-1}}$ is in $L_{p_1}(xW^{-1})$. By 3.5, each E' in π intersects only finitely many sets $x_E W^{-1}$, so f is locally in L_{p_1}. Choose $y_1, y_2, \ldots, y_k$ such that $V^{-1}V \subseteq \bigcup_{j=1}^{k} y_j W^{-1}$. Then $\bar{g} = (1/\delta)\sum_{j=1}^{k} g_{y_j} \in K^+(G)$. Since the hypothesis applies as well to right translates of g, we must have $(|f|^{p_1} * \bar{g})^{1/p_1}$ in $L_{p_2}(G)$. Choose z_E in E

such that

$$m(E)^{-1}\int_E\left[\left(|f|^{p_1} * \bar{g}\right)(x)\right]^{p_2/p_1} dm(x)$$

$$\geqslant \left[\left(|f|^{p_1} * \bar{g}\right)(z_E)\right]^{p_2/p_1} = \delta^{-p_2/p_1}\left[\sum_{j=1}^{k}\left(|f|^{p_1} * g_{y_j}\right)(z_E)\right]^{p_2/p_1}$$

$$\geqslant \left[\int_S |f(x)|^{p_1}\, dm(x)\right]^{p_2/p_1} \geqslant \left(\|f_E\|_{p_1}\right)^{p_2}$$

where $S = \bigcup_{j=1}^{k} z_E y_j W^{-1} \supset z_E V^{-1} V \supset E$. Thus $m(U)^{-1/p_2}\|\,|f|^{p_1} * \bar{g}\|_{p_2} \geqslant \|f\|_{\underline{p}}^{\pi}$.

The proof of the following is analogous.

Proposition 3.10. *The following are pairwise equivalent for* $1 \leqslant p_1 < \infty$.

(a) *f is in* $L^{\pi}_{(p_1,0)}$.

(b) $|f|^{p_1} * g$ *is in* $C_\infty(G)$ *for some* $g \not\equiv 0$ *in* $K^+(G)$.

(c) $|f|^{p_1} * g$ *is in* $C_\infty(G)$ *for all* g *in* $K^+(G)$.

The term "mixed-norm space" is used by Benedek and Panzone **[1]** for normed spaces of (equivalence classes of) functions on products of measure spaces. Our mixed norm spaces are equivalent to a special case of those defined by Benedek and Panzone. The next few propositions establish this, justifying our use of their term. We present them without proofs, which are straightforward.

Let $(X, \mathfrak{S}, \upsilon)$ and $(Y, \mathfrak{T}, \rho)$ be totally σ-finite measure spaces. A measurable function f on $(X \times Y, \mathfrak{S} \times \mathfrak{T}, \upsilon \times \rho)$ is said in **[1]** to be in $\underline{L}^{\underline{p}}(X \times Y)$ if $(\cdot, y)$ is in $L_{p_1}(X)$ for (ρ) almost all y in Y and $\|f(\cdot, y)\|_{p_1}$ is in $L_{p_2}(Y)$. Then $\underline{L}^{\underline{p}}$ is a Banach space in the obvious way.

Emerson and Greenleaf **[5]** prove the existence of a relatively compact Borel neighborhood S of the identity and a discrete set $F \subset G$ such that $\pi = \{xS: x \in F\}$ is a partition of G. (The countability hypothesis in the following is to avoid complications in defining product measures.)

Proposition 3.11. *With the above notation, if G is second countable it is Borel isomorphic with $S \times F$ in such a way that an isomorphic isomorphism is induced between* $\underline{L}^{\underline{p}}(S \times F)$ *and* $L_{\underline{p}}^{\pi}(G)$.

For $\bar{1} \leqslant \underline{p} < \bar{\infty}$ this shows $L_{\underline{p}}^{\pi}(G)$ is the norm completion of $L_{p_1}(S) \otimes l_{p_2}(F)$ which it contains in an obvious way.

We will now characterize compact subsets of $L_{\underline{p}}^{\pi}$ for $\bar{1} \leqslant \underline{p} < \bar{\infty}$. We first show the set of translation operators is uniformly bounded on $L_{\underline{p}}^{\pi}$.

Lemma 3.12. *Let π and F be as in 3.4. Then for $\bar{1} \leqslant \underline{p} < \bar{\infty}$, if f is in $L_{\underline{p}}^{\pi}(G)$ then* $\sup_{x\in G}\|{}_x f\|_{\underline{p}}^{\pi} \leqslant n_F(V, V)\|f\|_{\underline{p}}^{\pi}$.

Proof. Define $S(xE) = \{E' \in \pi: xE \cap E' \neq \varnothing\}$, $T(xE) = \{E' \in \pi: E \in S(xE')\}$. By 3.4, neither $S(xE)$ nor $T(xE)$ has more than $n_F(V, V)$ elements, so by

2.1 we have

$$\left[\int_E |_x f(y)|^{p_1}\, dm(y)\right]^{p_2/p_1} \leqslant \left[\sum_{E'\in S(xE)} \int_{E'} |f(y)|^{p_1}\, dm(y)\right]^{p_2/p_1}$$

$$\leqslant \left[\sum_{E'\in S(xE)} \left[\int_{E'} |f(y)|^{p_1}\, dm(y)\right]^{1/p_1}\right]^{p_2}$$

$$\leqslant (n_F(V, V))^{p_2-1} \sum_{E'\in S(xE)} \left[\int_{E'} |f(y)|^{p_1}\, dm(y)\right]^{p_2/p_1};$$

$$(\|_x f\|_p^\pi)^{p_2} \leqslant [n_F(V, V)]^{p_2-1} \sum_{E\in\pi} \sum_{E'\in S(xE)} \left[\int_{E'} |f(y)|^{p_1}\, dm(y)\right]^{p_2/p_1}$$

$$= [n_F(V, V)]^{p_2-1} \sum_{E'\in\pi} \sum_{E\in T(xE')} \left[\int_{E'} |f(y)|^{p_1}\, dm(y)\right]^{p_2/p_1}$$

$$\leqslant [n_F(V, V)]^{p_2} \sum_{E'\in\pi} \left[\int_{E'} |f(y)|^{p_1}\, dm(y)\right]^{p_2/p_1} = [n_F(V, V)]^{p_2}(\|f\|_p^\pi)^{p_2}.$$

PROPOSITION 3.13. *For $\underline{1} \leqslant \underline{p} < \overline{\infty}$, a subset H of L_p^π is relative compact iff*

(a) *H is bounded.*

(b) *The family of maps of G into L_p^π defined for each f in H by $x \to {}_x f$ is equicontinuous at the identity.*

(c) *For all $\varepsilon > 0$, there is a set J of compact complement such that $\|f_J\|_p^\pi < \varepsilon$ for all f in H.*

PROOF (NOTATION OF 3.12). Necessity of (a) is clear by continuity of the norm. Let $k \in K(G)$ have support S. By 1.4, at most $n_F(V, S)$ of π intersect S, so $\|k\|_p^\pi \leqslant n_F(V, S)^{1/p_2} m(V)^{1/p_1}\|k\|_\infty$. If H is relatively compact, for any $\varepsilon > 0$ there is a finite subset $\{k_1, k_2, \ldots, k_n\}$ of $K(G)$ such that, for any f in H, $\|f - k_i\|_p^\pi \leqslant \varepsilon/2(n_F(V, V) + 1)$ for some i. For x in some neighborhood W of the identity, we have $\|_x k_i - k_i\|_p^\pi < \varepsilon/2$ for all i, so using 3.12 we have

$$\|_x f - f\|_p^\pi \leqslant \|_x f - {}_x k_i\|_p^\pi + \|_x k_i - k_i\|_p^\pi + \|f - k_i\|_p^\pi$$

$$\leqslant [n_F(V, V) + 1]\|f - k_i\|_p^\pi + \|_x k_i - k_i\|_p^\pi < \varepsilon$$

and (b) is proved. The necessity of (c) follows from choosing $\{k_i\}$ as above and noting that all the k_i vanish outside a fixed compact set.

The proposition is known for $L_p(G)$. We use this to prove sufficiency. From (c), for any $\varepsilon > 0$ we can find $Q = G \sim J$ such that, for f in H,

$$\|f_J\|_p^\pi < \varepsilon/2(n_F(V, V) + 1).$$

From (b) we can find a neighborhood W of the identity such that $\|_x f - f\|_p^\pi < \varepsilon/2$ for all x in W and f in H. From (a) we have $\|f_Q\|_p^\pi = \|f - f_J\|_p^\pi \leqslant \|f\|_p^\pi + \|f_J\|_p^\pi < M + \varepsilon/2$ for all f in H. Consider the set H_Q of all functions f_Q, f in H, as a subset of $L_p^\pi(Q)$. By our last remark, (a) applies to H_Q. But for x in W we have

$$\|{}_xf_Q - f_Q\|_p^\pi = \|({}_xf - {}_xf_J - (f - f_J))\|_p^\pi \leq \|{}_xf - f\|_p^\pi + \|{}_xf_J - f_J\|_p^\pi$$
$$< \varepsilon/2 + \|{}_xf_J\|_p^\pi + \|f_J\|_p^\pi \leq \varepsilon/2 + (n_F(V, V) + 1)\|f_J\|_p^\pi < \varepsilon.$$

Thus (b) also applies to H_Q. By definition, (c) applies to the H_Q. On the compact set Q, however, the norm $\|\cdot\|_p^\pi$ and the usual $L_{p_1}(Q)$ norm are equivalent (by 2.1). Thus (a), (b), (c) hold for H_Q regarded as a subset of $L_{p_1}(Q)$ and H_Q is relatively compact in $L_{p_1}(Q)$. Thus we can find $\{h_1, h_2, \ldots, h_n\}$ in H_Q such that for all f in H, $\|f_Q - h_i\|_p^\pi \leq \varepsilon/2$ for some h_i (again, by norm equivalence). Thus $\|f - h_i\|_p^\pi = \|f_J + f_Q - h_i\|_p^\pi \leq \|f_J\|_p^\pi + \|f_Q - h_i\|_p^\pi < \varepsilon$ and H is totally bounded, hence relatively compact, in $L_p^\pi(G)$.

Our results on compactness of PC operators require a mixed-norm space different from those discussed above. The remainder of the section develops these spaces.

Let Λ_U denote the projection operator taking the measurable function f to f_U supported on the measurable set U and let T_x carry f to f_x; note that $T_{x^{-1}}\Lambda_U T_x = \Lambda_{Ux}$.

DEFINITION 3.14. For $1 \leq p, q \leq \infty$, we say f is a local p-q multiplier if, for every pair (E, H) of relatively compact sets of positive measure, $\Lambda_H C_f \Lambda_E$: $L_p(G) \to L_q(G)$ (where $\Lambda_H C_f \Lambda_E(g) = (f * g_E)_H$) is defined and bounded. We denote the bound by $\|\Lambda_H C_f \Lambda_E\|_{p,q}$.

LEMMA 3.15. *With the above notation, if f is a local p-q multiplier where $1 \leq p, q < \infty$ then* $\|\Lambda_H C_f \Lambda_E\|_{p,q} = \|\Lambda_{Hx} C_f \Lambda_{Ex}\|_{p,q}$.

PROOF. C_f commutes with right translation, so

$$\Lambda_{Hx} C_f \Lambda_{Ex} = T_{x^{-1}}\Lambda_H T_x C_f T_{x^{-1}}\Lambda_E T_x = T_{x^{-1}}\Lambda_H C_f \Lambda_E T_x.$$

But writing $\Delta(x)$ for the modular function on G, we have

$$(\|T_{x^{-1}}\Lambda_H C_f \Lambda_E T_x h\|_q)^q = \int_G |(\Lambda_H C_f \Lambda_E T_x h)(yx^{-1})|^q \, dm(y)$$
$$= \Delta(x)\int_G |(\Lambda_H C_f \Lambda_E T_x h)(u)|^q \, dm(u) \leq \Delta(x)(\|\Lambda_H C_f \Lambda_E\|_{p,q})^q(\|T_x h\|_p)^q$$
$$= \Delta(x)\Delta(x^{-1})(\|\Lambda_H C_f \Lambda_E\|_{p,q})^q(\|h\|_p)^q.$$

Thus $\|T_{x^{-1}}\Lambda_H C_f \Lambda_E T_x\|_{p,q} \leq \|\Lambda_H C_f \Lambda_E\|_{p,q}$ and symmetry, replacing x by x^{-1}, yields equality. For the infinite index case, the same argument yields:

LEMMA 3.16. (a) $\|\Lambda_{Hx} C_f \Lambda_{Ex}\|_{\infty,\infty} = \|\Lambda_H C_f \Lambda_E\|_{\infty,\infty}$.
(b) $\|\Lambda_{Hx} C_f \Lambda_{Ex}\|_{p,\infty} = [\Delta(x)]^{-1/p}\|\Lambda_H C_f \Lambda_E\|_{p,\infty}$ *for* $1 \leq p < \infty$.
(c) $\|\Lambda_{Hx} C_f \Lambda_{Ex}\|_{\infty,q} = [\Delta(x)]^{1/q}\|\Lambda_H C_f \Lambda_E\|_{\infty,q}$ *for* $1 \leq q < \infty$.

An analog of Lemma 3.2 need not hold for right translates. There is no bound on the (finite) number of members of a uniform partition which can intersect a right translate of a member. This problem can be avoided if G has a relatively compact invariant neighborhood of the identity, i.e., G is an (IN) group.

DEFINITION 3.17. Let G be an (IN) group, π a uniform partition, and $1 \leqslant p, q \leqslant \infty$, $1 \leqslant s < \infty$. We say f is a local p-q multiplier of class s relative to π if, for some H in π, $\Sigma_{E\in\pi}(\|\Lambda_H C_f \Lambda_E\|_{p,q})^s < \infty$, i.e., $\|\Lambda_H C_f \Lambda_E\|_{p,q}$ (regarded as a function of E in π) is in $l^s(\pi)$. We denote the space of such f by $LM^{\pi}_{p,q;s}$ and denote the norm in $l^s(\pi)$ by $\|f\|^{\pi,H}_{p,q;s}$. Similarly $LM^{\pi}_{p,q;\infty}$ and $LM^{\pi}_{p,q;0}$ are defined in terms of membership in $l^\infty(\pi)$ and $c_0(\pi)$, respectively.

PROPOSITION 3.18. *With the above notation, $LM^{\pi}_{p,q;s}$ is independent of the choice of H; the norms obtained for different choices of H are equivalent.*

PROOF. We give the case $1 \leqslant s < \infty$. Proofs for $s = 0, \infty$ are similar. As usual, π and F are as in Proposition 3.4.

Let N be a relatively compact invariant neighborhood of the identity, H_1, H_2 members of π. There is a finite set A of cardinality $|A|$ such that $H_2 \subset \bigcup_{x\in A} H_1 x$. For $E \in \pi$, x in A we write

$$S(E,x) = \{E' \in \pi: E' \cap Ex^{-1} \neq \varnothing\},$$
$$T(E,x) = \{E' \in \pi: E \in S(E',x)\}.$$

There are $\{x_1, x_2, \ldots, x_k\}$ such that $E \subset \bigcup_i x_i N$ and if $E' \in S(E,x)$ then $E' \cap \bigcup_i x_i N x^{-1} = \bigcup_i (E' \cap x_i x^{-1} N) \neq \varnothing$. By 3.4, $E' \cap x_i x^{-1} N \neq \varnothing$ can hold for no more than $n_F(V,N)$ sets E' in π; the cardinality of $S(E,x)$ is at most $kn_F(V,N)$. Since every left translate of V can be covered by the same number of left translates of N, k can be independent of E. Thus $kn_F(V,N)$ is a bound on the cardinality of all the $S(E,x)$. A similar argument shows it is such a bound for the $T(E,x)$. Now

$$\|\Lambda_{H_2} C_f \Lambda_E\|_{p,q} \leqslant \sum_{x\in A} \|\Lambda_{H_1 x} C_f \Lambda_E\|_{p,q} = \sum_{x\in A} \|\Lambda_{H_1} C_f \Lambda_{Ex^{-1}}\|_{p,q}$$

by 3.15 and 3.16. (Recall that (IN) implies unimodular.) But

$$\sum_{x\in A} \|\Lambda_{H_1} C_f \Lambda_{Ex^{-1}}\|_{p,q} \leqslant \sum_{x\in A} \sum_{E'\in S(E,x)} \|\Lambda_{H_1} C_f \Lambda_{E'}\|_{p,q}$$

since $\Lambda_{Ex^{-1}} = \Sigma_{E'\in S(E,x)} \Lambda_{E'} \Lambda_{Ex^{-1}}$. Thus, using 2.1,

$$\begin{aligned}
\sum_{E\in\pi} \left(\|\Lambda_{H_2} C_f \Lambda_E\|_{p,q}\right)^s &\leqslant \sum_{E\in\pi} \left[\sum_{x\in A} \sum_{E'\in S(E,x)} \left(\|\Lambda_{H_1} C_f \Lambda_{E'}\|_{p,q}\right)\right]^s \\
&\leqslant \sum_{E\in\pi} \left[kn_F(V,N)|A|\right]^{s-1} \sum_{x\in A} \left[\sum_{E'\in S(E,x)} \left(\|\Lambda_{H_1} C_f \Lambda_E\|_{p,q}\right)\right]^s \\
&= \sum_{E'\in\pi} \sum_{x\in A} \sum_{E\in T(E',x)} \left[|A| kn_F(V,N)\right]^{s-1} \left(\|\Lambda_{H_1} C_f \Lambda_{E'}\|_{p,q}\right)^s \\
&\leqslant \left(|A| kn_F(V,N)\right)^s \sum_{E'\in\pi} \left(\|\Lambda_{H_1} C_f \Lambda_{E'}\|_{p,q}\right)^s
\end{aligned}$$

and by interchange of H_1 and H_2, the norms are equivalent and the spaces $LM_{p,q;s}$ are independent of the choice of the set H, in their definition. Moreover, the equivalence class of norms is independent of π.

4. Convolution and mixed-norm spaces. If G is unimodular, $1 \leqslant p,\ q \leqslant \infty$, $1/p + 1/q \geqslant 1$ and $1/r = 1/p + 1/q - 1$, then $L_p(G) * L_q(G) \subset L_r(G)$ and $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$ (Young's inequality). Attempting to extend Young's inequality we encounter products of sets from π and can find no bound on the number of members of π intersecting such a product. As in the discussion of local multipliers, the appropriate assumption appears to be that G is (IN). *For the rest of this paper, G will be an (IN) group and N will denote a fixed symmetric compact invariant neighborhood of the identity.*

LEMMA 4.1. *Let π and F be as in* 3.4. *For E_1, E_2 in π, define*

$$S(E_1, E_2) = \{E \in \pi: E \cap E_2 E_1^{-1} \neq \varnothing\},$$
$$T(E_1, E_2) = \{E \in \pi: E_1 \in S(E, E_2)\},$$
$$W(E_1, E_2) = \{E \in \pi: E_1 \in S(E_2, E)\}.$$

There is an integer $\Gamma(\pi)$ which uniformly bounds S, T and W.

PROOF. Let A be a finite set of cardinality $|A|$ such that $V \subset \bigcup_{y \in A} yN$. Then if $E \in S(E_1, E_2)$, $\varnothing \neq x_E V \cap x_{E_2} V V^{-1} x_{E_1}^{-1} \subset \bigcup_{y \in A} \bigcup_{z \in A} [x_E V \cap x_{E_2} y z^{-1} x_{E_1}^{-1} N^2]$ by the invariance and symmetry of N and N^2. By 3.2, there are at most $n_F(V, N^2) x_E$ for which each set of the union is nonempty. Thus $S(E_1, E_2)$ has at most $|A|^2 n_F(V, N^2)$ members. Since $E \in T(E_1, E_2)$ (or $W(E_1, E_2)$) iff $E \cap E_1^{-1} E_2 \neq \varnothing$ (or $E \cap E_1 E_2 \neq \varnothing$), similar computations show we can choose $\Gamma(\pi) = |A|^2 n_F(V, N^2)$ in all cases.

THEOREM 4.2 (CF. [**18**]). *Let $f \in L_{\underline{p}}^{\pi}$, $g \in L_{\underline{q}}^{\pi}$, $\bar{1} \leqslant \underline{p}, \underline{q} \leqslant \overline{\infty}$, $1/p_2 + 1/q_2 \geqslant 1$; then*

(a) *$f * g$ exists* (*and is finite*) *locally a.e. and lies in $L_{\underline{r}}^{\pi}$ where*

$$1/r_1 = \max\{0, 1/p_1 + 1/q_1 - 1\}, \qquad 1/r_2 = 1/p_2 + 1/q_2 - 1.$$

(b) $\|f * g\|_{\underline{r}}^{\pi} \leqslant \Gamma(\pi) \|f\|_{\underline{p}}^{\pi} \|g\|_{\underline{q}}^{\pi}$.

PROOF. Suppose $p_2, q_2, r_2 \neq \infty$; then by Young's inequality
(i) $\|f_{E_1} * g_{E_2}\|_{r_1} \leqslant \|f_{E_1}\|_{p_1} \|g_{E_2}\|_{q_1} < \infty$ for any E_1, E_2 in π, and
(ii) $(f_{E_1} * g_{E_2})_{E_3} = 0$ if $E_1 \notin S(E_2, E_3)$. Now

$$\begin{aligned}\sum_{E_3 \in \pi} \sum_{E_2 \in \pi} \sum_{E_1 \in S(E_2, E_3)} &\left(\|f_{E_1}\|_{p_1}\right)^{p_2} \left(\|g_{E_2}\|_{q_1}\right)^{q_2} \\ &= \sum_{E_2 \in \pi} \left(\|g_{E_2}\|_{q_1}\right)^{q_2} \sum_{E_1 \in \pi} \sum_{E_3 \in U(E_1, E_2)} \left(\|f_{E_1}\|_{p_1}\right)^{p_2} \\ &\leqslant \Gamma(\pi) \left(\|f\|_{\underline{p}}^{\pi}\right)^{p_2} \left(\|g\|_{\underline{q}}^{\pi}\right)^{q_2} < \infty.\end{aligned}$$

Thus for each $E_3 \in \pi$,

$$\sum_{E_2 \in \pi} \sum_{E_1 \in S(E_2, E_3)} \left(\|f_{E_1}\|_{p_1}\right)^{p_2} \left(\|g_{E_2}\|_{q_1}\right)^{q_2} < \infty.$$

By the generalized (multiproduct) Hölder inequality

$$\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|f_{E_1}\|_{p_1}\right)\left(\|g_{E_2}\|_{q_1}\right)$$

$$\leq\left[\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|f_{E_1}\|_{p_1}\right)^{p_2}\left(\|g_{E_2}\|_{q_1}\right)\right]^{1/r_2}$$

$$\cdot\left[\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|f_{E_1}\|_{p_1}\right)^{p_2}\right]^{(q_2-1)/q_2}$$

$$\cdot\left[\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|g_{E_2}\|_{q_1}\right)^{q_2}\right]^{(p_2-1)/p_2}$$

$$=\left[\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|f_{E_1}\|_{p_1}\right)^{p_2}\left(\|g_{E_2}\|_{q_1}\right)^{q_2}\right]^{1/r_2}$$

$$\cdot\left[\sum_{E_1\in\pi}\sum_{E_2\in T(E_1,E_3)}\left(\|f_{E_1}\|_{p_1}\right)^{p_2}\right]^{(q_2-1)/q_2}$$

$$\cdot\left[\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|g_{E_2}\|_{q_1}\right)^{q_2}\right]^{(p_2-1)/p_2}$$

$$\leq\left[\Gamma(\pi)\right]^{(2-1/p_2-1/q_2)}\left(\|f\|_{\underline{p}}^{\pi}\right)^{p_2(q_2-1)/q_2}\left(\|g\|_{\underline{q}}^{\pi}\right)^{q_2(p_2-1)/p_2}$$

$$\cdot\left[\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|f_{E_1}\|_{p_1}\right)^{p_2}\left(\|g_{E_2}\|_{q_1}\right)^{q_2}\right]^{1/r_2}.$$

Thus for any E_3, by (ii)

$$\left(\|(f*g)_{E_3}\|_{r_1}\right)^{r_2}\leq\left[\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\|f_{E_1}*g_{E_2}\|_{r_1}\right]^{r_2}$$

$$\leq\left[\Gamma(\pi)\right]^{(2-1/p_2-1/q_2)r_2}\left(\|f\|_{\underline{p}}^{\pi}\right)^{r_2p_2(q_2-1)/q_2}\left(\|g\|_{\underline{q}}^{\pi}\right)^{r_2q_2(p_2-1)/p_2}$$

$$\cdot\left[\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|f_{E_1}\|_{p_1}\right)^{p_2}\left(\|g_{E_2}\|_{q_1}\right)^{q_2}\right],$$

$$\left(\|f*g\|_{\underline{r}}^{\pi}\right)^{r_2}=\sum_{E_3\in\pi}\left(\|(f*g)_{E_3}\|_{r_1}\right)^{r_2}$$

$$\leq\left[\Gamma(\pi)\right]^{(2-1/p_2-1/q_2)r_2}\left(\|f\|_{\underline{p}}^{\pi}\right)^{r_2p_2(q_2-1)/q_2}\left(\|g\|_{\underline{q}}^{\pi}\right)^{r_2q_2(p_2-1)/p_2}$$

$$\cdot\left[\sum_{E_3\in\pi}\sum_{E_2\in\pi}\sum_{E_1\in S(E_2,E_3)}\left(\|f_{E_1}\|_{p_1}\right)^{p_2}\left(\|g_{E_2}\|_{q_1}\right)\right]^{q_2}$$

$$\leq\left[\Gamma(\pi)\right]^{(2-1/p_2-1/q_2)r_2+1}\left(\|f\|_{\underline{p}}^{\pi}\right)^{p_2+r_2p_2(q_2-1)/q_2}\left(\|g\|_{\underline{q}}^{\pi}\right)^{q_2-r_2q_2(p_2-1)/p_2}$$

$$=\left[\Gamma(\pi)\right]^{r_2}\left(\|f\|_{\underline{p}}^{\pi}\right)^{r_2}\left(\|g\|_{\underline{q}}^{\pi}\right)^{r_2},$$

$$\|f*g\|_{\underline{r}}^{\pi}\leq\Gamma(\pi)\|f\|_{\underline{p}}^{\pi}\|g\|_{\underline{q}}^{\pi}$$

which proves both (a) and (b) for this case.

Suppose $p_2, q_2 \neq \infty$ but $r_2 = \infty$ (so $1/p_2 + 1/q_2 = 1$). By 4.1 and 2.1

$$\begin{aligned}\sum_{E_2 \in \pi}\Big(\sum_{E_1 \in S(E_2,E_3)} \|f_{E_1}\|_{p_1}\Big)^{p_2} &\leqslant [\Gamma(\pi)]^{p_2/q_2} \sum_{E_2 \in \pi} \sum_{E_1 \in S(E_2,E_3)} \big(\|f_{E_1}\|_{p_1}\big)^{p_2}\\ &= [\Gamma(\pi)]^{p_2/q_2} \sum_{E_1 \in \pi} \sum_{E_2 \in T(E_1,E_3)} \big(\|f_{E_1}\|_{p_1}\big)^{p_2}\\ &\leqslant [\Gamma(\pi)]^{(p_2/q_2)+1} \sum_{E_1 \in \pi} \big(\|f_{E_1}\|_{p_1}\big)^{p_2}\\ &= [\Gamma(\pi)]^{p_2}(\|f\|_{\underline{p}}^{\pi})^{p_2}.\end{aligned}$$

Computing as before

$$\begin{aligned}\|(f * h)_{E_3}\|_{r_1} &\leqslant \sum_{E_2 \in \pi} \sum_{E_1 \in S(E_2,E_3)} \|f_{E_1}\|_{p_1}\|g_{E_2}\|_{q_1}\\ &= \sum_{E_2 \in \pi} \|g_{E_2}\|_{q_1} \sum_{E_1 \in S(E_2,E_3)} \|f_{E_1}\|_{p_1}\\ &\leqslant \Big(\sum_{E_2 \in \pi}\big(\|g_{E_2}\|_{q_1}\big)^{q_2}\Big)^{1/q_2}\Bigg(\sum_{E_2 \in \pi}\Big(\sum_{E_1 \in S(E_2,E_3)} \|f_{E_1}\|_{p_1}\Big)^{p_2}\Bigg)^{1/p_2}\\ &\leqslant \Gamma(\pi)\|f\|_{\underline{p}}^{\pi}\|g\|_{\underline{q}}^{\pi}.\end{aligned}$$

Thus for this case we also have $\|f * g\|_{\underline{r}}^{\pi} = \sup_{E_3 \in \pi}\|(f * g)_{E_3}\|_{r_1} \leqslant \Gamma(\pi)\|f\|_{\underline{p}}^{\pi}\|g\|_{\underline{q}}^{\pi}$.

Now suppose $p_2 = \infty$ (so $q_2 = 1$, $r_2 = \infty$). Then $\sup_{E_1 \in \pi}\|f_{E_1}\|_{p_1} < \infty$, $\sum_{E_2 \in \pi}\|g_{E_2}\|_{q_1} < \infty$,

$$\|(f * g)_{E_3}\|_{r_1} \leqslant \sum_{E_2 \in \pi} \sum_{E_1 \in S(E_1,E_2)} \|f_{E_1}\|_{p_1}\|g_{E_2}\|_{q_1} \leqslant \|f\|_{\underline{p}}^{\pi}\Gamma(\pi) \sum_{E_2 \in \pi} \|g_{E_2}\|_{q_1},$$

$$\|f * g\|_{\underline{r}}^{\pi} \leqslant \Gamma(\pi)\|f\|_{\underline{p}}^{\pi}\|g\|_{\underline{g}}^{\pi}.$$

A similar calculation yields the case $q_2 = \infty$.

REMARKS. (i) By 3.13(b) translation is continuous, so from 4.2 $f * g$ is uniformly continuous if $f \in L_{\underline{p}}^{\pi}$, $g \in L_{\underline{p}'}^{\pi}$, $\underline{1} < \underline{p} < \underline{\infty}$, just as for usual L_p's.

(ii) It follows from 4.2 that the $L_{(p,1)}^{\pi}(G)$ are group algebras (i.e., closed under convolution) for all $p \geqslant 1$.

(iii) J. Stewart has proved [**16**, §4, Lemma 1] for G abelian with dual $G^\wedge$, that for every compact set K in $G^\wedge$, there is f in $L_{(2,1)}^{\pi}(G) \cap L_1(G)$ such that its Fourier transform $f^\wedge \equiv 1$ on K and has compact support. (By 3.7(c) we know $L_{(2,1)}^{\pi} \subset L_1$, so Stewart's condition is redundant.) By 4.2, $f * f$ is in $L_{(\infty,1)}^{\pi}(G) \subset L_1(G) \cap L_\infty(G)$ and its Fourier transform $(f^\wedge)^2$ has the desired properties. Thus there is actually a *bounded* L_1 function with such a Fourier transform.

LEMMA 4.3. *Suppose $1 \leqslant p < \infty$. For every nonnegative sequence α not in $l_p(Z^+)$ there is a nonnegative sequence β in $l_{p'}(Z^+)$ such that $\sum_{n=1}^{\infty} \alpha_n\beta_n = +\infty$.*

PROOF. If $p = 1$, set $\beta_n \equiv 1$. If $p = \infty$, choose a subsequence such that $\alpha_k > k$. Let the corresponding β_k be $1/k^2$ and let β vanish elsewhere. Then clearly β is in $l_1(Z^+)$ and the condition holds. Finally, suppose $1 < p < \infty$. By assumption,

$\Sigma_{n=1}^{\infty}(\alpha_n)^p = +\infty$, so if $D_n = (\alpha_1)^p + (\alpha_2)^p + \cdots + (\alpha_n)^p$, Dini's theorem on sequences (see [11]) says $\Sigma_{n=1}^{\infty}(\alpha_n)^p/(D_n)^\gamma$ is infinite for $\gamma \leq 1$ and finite for $\gamma > 1$. Let $\beta_n = \alpha_n^{p-1}/D_n$. Then by Dini's theorem

$$\sum_{n=1}^{\infty} \beta_n^{p'} = \sum_{n=1}^{\infty} \frac{\alpha_n^{(p-1)p'}}{D_n^{p'}} = \sum_{n=1}^{\infty} \frac{\alpha_n^p}{D_n^{p'}} < \infty,$$

$$\sum_{n=1}^{\infty} \alpha_n\beta_n = \sum_{n=1}^{\infty} \frac{\alpha_n^p}{D^n} = +\infty.$$

THEOREM 4.4. *For f locally measurable and $\bar{1} \leq \underline{p} \leq \overline{\infty}$ the following are equivalent.*

(a) *$(f * g)$ exists (and is finite) locally a.e. for all g in $L_{\underline{p}}^\pi$.*

(b) *$(g * f)$ exists (and is finite) locally a.e. for all g in $L_{\underline{p}}^\pi$.*

(c) *f is in $L_{(1,p_2')}^\pi$.*

PROOF. By 3.8, we can assume π in N-N^2 uniform. By symmetry and invariance of N, $\pi^{-1} = \{E^{-1}: E \in \pi\}$ is also an N-N^2 uniform partition. By 3.8, the map $f \to f^\sim$, $f^\sim(x) = f(x^{-1})$, leaves all the spaces $L_{\underline{p}}^\pi$ invariant. Since $f * g = (g^\sim * f^\sim)^\sim$ it follows that (a) and (b) are equivalent.

By 4.2, (c) implies (a) and, in fact, $f * g$ is in $L_{(p_1,\infty)}^\pi$. We need only prove (a) implies (c). Let E be in π, W a relatively compact neighborhood of the identity, and g the characteristic function of $E^{-1}W^{-1}$. Since $g \in L_{\underline{p}}^\pi$, (a) implies that $f * g$ exists for some x in W^{-1};

$$\infty > \int_G |f(y)|g(y^{-1}x)\, dm(y) = \int_{xWE} |f(y)|\, dm(y) \geq \int_E |f(y)|\, dm(y);$$

f is locally integrable.

Choose a maximal family of pairwise disjoint left translates $\{y_jN^2: j \in J\}$ of N^2 and let H be a finite family of elements of G such that $N^4 \subset \bigcup_{z\in H} zN$. By maximality $G = \bigcup_{j\in J} \bigcup_{z\in H} y_jzN = \bigcup_{z\in H}[\bigcup_{j\in J} y_jN]z$. If f is not in $L_{(1,p_2')}^\pi$, there must be z_0 in H such that the restriction of f to $\bigcup_{j\in J} y_jNz_0$ is not in $L_{(1,p_2')}^\pi$, so there must be some subsequence of the y_j's indexed by Z^+, such that the restriction of f to $\bigcup_{n=1}^{\infty} y_nNz_0$ is not in $L_{(1,p_2')}^\pi$. Taking α in 4.3 to be $\{\|f_{y_nz_0N}\|_1\}$, there is nonnegative β in $l_{p_2}(Z^+)$ such that $\Sigma_{n=1}^{\infty} \alpha_n\beta_n = \infty$. Let g be a function defined to be identically β_n on the set $z_0^{-1}y_n^{-1}N^2$ are zero elsewhere. (Disjointness of $z_0^{-1}y_n^{-1}N^2$ follows from that of y_jN^2.) Clearly $g \in L_{\underline{p}}^\pi$ and if $x \in N$ we have

$$\int_G |f(y)g(y^{-1}x)|\, dm(y) \geq \sum_{n=1}^{\infty} \int_{y_nNz_0} |f(y)|g(y^{-1}x)\, dm(y).$$

If $y \in y_nNz_0$, $y^{-1}x \in z_0^{-1}y_n^{-1}N^2$, so $g(y^{-1}x) = \beta_n$. Thus, for all x in N,

$$\int_G |f(y)g(y^{-1}x)|\, dm(y) \geq \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty.$$

Thus $f * g(x)$ is not defined on N, a compact set of positive measure; assuming f is not in $L_{(1,p_2')}^\pi$ contradicts (a).

N. Rickert proved [4] that for $1 \leqslant p, q \leqslant \infty$, $1/p + 1/q < 1$ on a locally compact group G which is not compact there must be f in $L_p(G)$ and g in $L_q(G)$ with $f * g$ undefined on a set of positive measure. Theorem 4.4 gives a precise characterization for (IN) groups and show it to be the rule, rather than the exception. For p, q as above, we say $f \in L_p(G)$ has a divergent q-convolution if there is some $g \in L_q(G)$ such that $f * g$ is undefined on a set of positive measure. Immediately from 4.4 we have:

PROPOSITION 4.5. *If $1 \leqslant p, q \leqslant \infty$, $1/p + 1/q < 1$ and $f \in L_p(G)$, f has a divergent q-convolution iff f is not in $L^{\pi}_{(1,q')}(G)$.*

COROLLARY 4.6. *The set of f in $L_p(G)$ having a divergent q-convolution* ($1 \leqslant p, q \leqslant \infty$, $1/p + 1/q < 1$) *is of second category in $L_p(G)$.*

PROOF. By 4.5 we need only show that $X = L_p \cap L^{\pi}_{(1,q')}$ is one of the first category. With norm $N(f) = \|f\|_p + \|f\|^{\pi}_{(1,q')}$, X is a Banach space and the natural injection of X into L_p is continuous. By 2.11 of [15], since X is not all of L_p it must be first category.

5. Product-convolution operators. Let $\mathfrak{M}(G)$ be the locally measurable functions, with locally a.e. equal functions identified. For h in $\mathfrak{M}(G)$, p_h denotes pointwise multiplication by h ($(P_h g)(x) = h(x)g(x)$). For f in $\mathfrak{M}(G)$, C_f denotes the operator having as domain the g in $\mathfrak{M}(G)$ such that $f * g$ exists (and is finite) locally a.e. and which maps g to $f * g$. A product-convolution (PC) operator is a composition $P_h C_f$. We will be concerned with cases where the domain of the PC operator contains, and is restricted to, one of the mixed-norm spaces and the range of this restriction lies in another. In this section we study conditions for the operator to be bounded; in the next, conditions for it to be compact.

THEOREM 5.1. *The operator $P_h C_f$, with $h \in L^{\pi}_{\underline{p}}$ and $f \in L^{\pi}_{\underline{r}}$, bounded from $L^{\pi}_{\underline{q}}$ to $L^{\pi}_{\underline{s}}$ where $\underline{1} \leqslant \underline{p}, \underline{q}, \underline{r}, \underline{s} \leqslant \underline{\infty}$, if*
(a) $1/\underline{p} + 1/\underline{r} \succ \underline{1} + 1/\underline{s} - 1/\underline{q}$ *and*
(b) $(s_1, r_2) \leqslant (p_1, q_2')$.
In this case, the bound satisfies

$$\|P_h C_f\|^{\pi}_{\underline{q},\underline{s}} \leqslant \Gamma(\pi)\|h\|^{\pi}_{\underline{p}}\|f\|^{\pi}_{\underline{r}},$$

($\Gamma(\pi)$ *is as in* 4.2).

PROOF. (b) implies $1 \leqslant 1/r_2 + 1/q_2$ so, defining $\underline{t}$ by

$$\frac{1}{t_1} = \max\left\{0, \frac{1}{r_1} + \frac{1}{q_1} - 1\right\}, \qquad \frac{1}{t_2} = \frac{1}{r_2} + \frac{1}{q_2} - 1,$$

(by 4.2) for any $g \in L^{\pi}_{\underline{q}}\ \|f * g\|^{\pi}_{\underline{t}} \leqslant \Gamma(\pi)\|f\|^{\pi}_{\underline{r}}\|g\|^{\pi}_{\underline{q}}$.

(a) and (b) imply $s_1 \leqslant p_1$ and $s_1 \leqslant t_1$. If $s_1 = \infty$, for any E in π,

$$\|[h \cdot (f * g)]_E\|_{s_1} \leqslant \|h_E\|_{p_1}\|(f * g)_E\|_{t_1}.$$

For $s_1 < \infty$, since (a) implies $1/p_1 + 1/t_1 \leqslant 1/s_1$ and $|h|^{s_1}_E \in L_{p_1/s_1}(E)$, $|f * g|^{s_1}_E \in L_{t_1/s_1}(E)$, Hölder's inequality yields

$$\|[|h|^{s_1}\cdot|(f*g)|^{s_1}]_E\|_1 \leqslant \|\,|h|_E^{s_1}\|_{p_1/s_1}\|\,|(f*g)|_E^{s_1}\|_{t_1/s_1}$$
$$= (\|h_E\|_{p_1}\|(f*g)_E\|_{t_1})^{s_1},$$

so the results hold in this case also. Thus if $s_2 = \infty$, by the above result,

$$\|h\cdot(f*g)\|_{\underline{s}}^{\pi} = \sup_{E\in\pi}\|[h\cdot(f*g)_E]\|_{s_1} \leqslant \sup_{E\in\pi}\|h_E\|_{p_1}\sup_{E\in\pi}\|(f*g)_E\|_{t_1}$$
$$\leqslant \|h\|_{\underline{p}}^{\pi}\|f*g\|_{\underline{t}}^{\pi} \leqslant \Gamma(\pi)\|h\|_{\underline{p}}^{\pi}\|f\|_{\underline{r}}^{\pi}\|g\|_{\underline{q}}^{\pi}.$$

If $s_2 < \infty$, let $k = \max\{1, p_2/s_2\}$, $n = \max\{1, t_2/s_2\}$. Since $1/p_2 + 1/t_2 \geqslant 1/s_2$, by (a) and the definition of t_2, $1/k + 1/n \geqslant 1$. Again by Hölder's inequality and the preceding result,

$$\sum_{E\in\pi}\|h_E\|_{p_1}^{s_2}\|(f*g)_E\|_{t_1}^{s_2} \leqslant \left[\sum_{E\in\pi}\|h_E\|_{p_1}^{s_2k}\right]^{1/k}\left[\sum_{E\in\pi}\|(f*g)_E\|_{t_1}^{s_2n}\right]^{1/n},$$

so

$$\|h\cdot(f*g)\|_{\underline{s}}^{\pi} \leqslant \left[\sum_{E\in\pi}(\|h_E\|_{p_1})^{s_2}(\|(f*g)_E\|_{t_1})^{s_2}\right]^{1/s_2}$$
$$\leqslant \left[\sum_{E\in\pi}(\|h_E\|_{p_1})^{s_2k}\right]^{1/s_2k}\left[\sum_{E\in\pi}(\|(f*g)_E\|_{t_1})^{s_2n}\right]^{1/s_2n}$$
$$\leqslant \left(\sum_{E\in\pi}(\|h_E\|_{p_1})^{p_2}\right)^{1/p_2}\left(\sum_{E\in\pi}(\|(f*g)_E\|_{t_1})^{t_2}\right)^{1/t_2}$$
$$= \|h\|_{\underline{p}}^{\pi}\|f*g\|_{\underline{t}}^{\pi} \leqslant \Gamma(\pi)\|h\|_{\underline{p}}^{\pi}\|f\|_{\underline{r}}^{\pi}\|g\|_{\underline{q}}^{\pi}.$$

COROLLARY 5.2. *The operator* P_hC_f *with* $h \in L_{\underline{p}}^{\pi}$ *and* $f \in L_{\underline{r}}^{\pi}$ *is bounded on* $L_{\underline{q}}^{\pi}$, *where* $\overline{1} \leqslant \underline{p}, \underline{q}, \underline{r} \leqslant \overline{\infty}$, *provided*

(a) $1/\underline{p} + 1/\underline{r} \succ \overline{1}$ *and*

(b) $(q_1, r_2) \leqslant (p_1, q_2)$.

In the next section we require a strengthened version of 5.1 which is also of interest in its own right:

THEOREM 5.3. *Suppose for* $\overline{1} \leqslant \underline{p}, \underline{q}, \underline{r}, \underline{s} \leqslant \overline{\infty}$

(a) $1/\underline{p} + 1/\underline{r} \succ \overline{1} + 1/\underline{s} - 1/\underline{q}$,

(b) $(s_1, r_2) \leqslant (p_1, q_2')$,

(c) $1/r_2 + 1/q_2 \geqslant 1 + 1/s_2$,

(d) $(s_1, r_2) < (p_1, \infty)$.

Then, for any f *in* $L_{\underline{r}}^{\pi}$, $\varepsilon > 0$, $\theta > 0$, *there is* $\delta > 0$ *such that for any* h *in* $L_{\underline{p}}^{\pi}$ *for which* $\|h\|_{\underline{p}}^{\pi} \leqslant \theta$ *and* $\|h\|_{(1,\infty)}^{\pi} \leqslant \delta$, *then* $\|P_hC_f\|_{\underline{q},\underline{s}}^{\pi} < \varepsilon$.

PROOF. Since $r_2 < \infty$ choose f_0 a bounded function of compact support such that $\|f - f_0\|_{\underline{r}}^{\pi} < \varepsilon/2\theta\Gamma(\pi)$. For g in $L_{\underline{q}}^{\pi}$ we know by 4.2 that

$$\|f_0 * g\|_{(\infty,q_2)}^{\pi} \leqslant \Gamma(\pi)\|f_0\|_{(\infty,1)}^{\pi}\|g\|_{\underline{q}}^{\pi} \tag{1}$$

and

$$\|f_0 * g\|_{\underline{t}}^{\pi} \leq \Gamma(\pi)\|f_0\|_{\underline{r}}^{\pi}\|g\|_{\underline{q}}^{\pi}, \tag{2}$$

$$\frac{1}{t_1} = \min\left\{0, \frac{1}{r_1} + \frac{1}{q_1} - 1\right\}, \qquad \frac{1}{t_2} = \frac{1}{r_2} + \frac{1}{q_2} - 1.$$

We take π to be U-V uniform. The usual inequality for L_p norms on sets of finite measure yields

$$\|f_0 * g\|_{(p_1s_1/(p_1-s_1),t_2)}^{\pi} \leq \Gamma(\pi)\|f_0\|_{\underline{r}}^{\pi}\|g\|_{\underline{q}}^{\pi} m(V)^{(1/s_1-1/p_1-1/t_1)} \tag{3}$$

since $1/s_1 - 1/p_1 \geq 1/t_1$. Let $\alpha = \varepsilon/(4\Gamma(\pi)m(V)^{1/s_1-1/t_1}\|f_0\|_{\underline{r}}^{\pi})$ and choose $\delta > 0$ so that

$$(\delta/\alpha)^{(1/s_1-1/p_1)}\Gamma(\pi)\theta\|f_0\|_{(\infty,1)}^{\pi} < \varepsilon/4. \tag{4}$$

For any $h \in L_{\underline{p}}^{\pi}$ such that $\|h\|_{\underline{p}}^{\pi} \leq \theta$ and $\|h\|_{(1,\infty)}^{\pi} \leq \delta$ and each $E \in \pi$, define $S_\alpha = \{x: |h(x)| \geq \alpha\}$, $E_1 = E \cap S_\alpha$, $E_2 = E \sim E_1$. Then for $i = 1, 2$

$$\int_{E_i} |h(x)|^{s_1}|(f_0 * g)(x)|^{s_1}\, dm(x) \leq \|(h^{s_1})_{E_i}\|_{p_1/s_1}\|(f_0 * g)_{E_i}^{s_1}\|_{p_1/(p_1-s_1)}$$

so $\|[h \cdot (f_0 * g)]_{E_i}\|_{s_1} \leq \|h_{E_i}\|_{p_1}\|(f_0 * g)_{E_i}\|_{p_1s_1/(p_1-s_1)} = \gamma_i$. For $i = 1$,

$$\begin{aligned}\gamma_1 &\leq \|h_E\|_{p_1}\|(f_0 * g)_E\|_\infty[m(E_1)]^{(1/s_1-1/p_1)} \\ &\leq \|h_E\|_{p_1}\|(f_0 * g)_E\|_\infty\left(\frac{1}{\alpha}\|h\|_{(1,\infty)}^{\pi}\right)^{(1/s_1-1/p_1)}\end{aligned}$$

since $\alpha m(E_1) \leq \|h_E\|_1 \leq \|h\|_{(1,\infty)}^{\pi}$. Thus, if $s_2 < \infty$,

$$\sum_{E\in\pi}\left(\|[h(f_0 * g)]_{E_1}\|_{s_1}\right)^{s_2} \leq \left(\frac{1}{\alpha}\|h\|_{(1,\infty)}^{\pi}\right)^{(s_2/s_1-s_2/p_1)}\sum_{E\in\pi}(\|h_E\|_{p_1})^{s_2}(\|(f_0 * g)_E\|_\infty)^{s_2}.$$

But by (1), $\{\|(f_0 * g)_E\|_\infty\} \in l_{q_2}(\pi)$ and by (a)

$$\frac{s_2}{p_2} + \frac{s_2}{q_2} \geq 1 + s_2\left(1 - \frac{1}{r_2}\right) = 1 + \frac{s_2}{r_2'} > 1,$$

so, using Hölder's inequality,

$$\begin{aligned}\sum_{E\in\pi}(\|h_E\|_{p_1})^{s_2}(\|(f_0 * g)_E\|_\infty)^{s_2} &\leq \left(\sum_{E\in\pi}(\|h_E\|_{p_1})^{p_2}\right)^{s_2/p_2}\left(\sum_{E\in\pi}(\|(f_0 * g)_E\|_\infty)^{q_2}\right)^{s_2/q_2} \\ &= (\|h\|_{\underline{p}}^{\pi})^{s_2}(\|(f_0 * g)\|_{(\infty,q_2)}^{\pi})^{s_2} \\ &\leq (\|h\|_{\underline{p}}^{\pi})^{s_2}(\Gamma(\pi)\|f_0\|_{(\infty,1)}^{\pi}\|g\|_{\underline{q}}^{\pi})^{s_2} \quad \text{(by (1))}.\end{aligned}$$

Thus

$$\|h \cdot (f_0 * g)_{S_\alpha}\|_{\underline{s}} \leq \left(\frac{1}{\alpha}\|h\|_{(1,\infty)}^{\pi}\right)^{1/s_1-1/p_1}\|h\|_{\underline{p}}^{\pi}\Gamma(\pi)\|f_0\|_{(\infty,1)}^{\pi}\|g\|_{\underline{q}}^{\pi} < \frac{\varepsilon}{4}\|g\|_{\underline{q}}^{\pi}$$

(by (4)). (If $s_2 = \infty$ the same inequality follows by similar computations.)

For $i = 2$, $\gamma_2 \leq \alpha(m(V))^{1/p_1}\|(f_0 * g)_E\|_{p_1s_1/(p_1-s_1)}$ and if $s_2 < \infty$ we have

$$\begin{aligned}\|(h \cdot (f_0 * g))_{G\sim S_\alpha}\|_{\underline{s}}^{\pi} &\leq \alpha(m(V))^{1/p_1}\left[\sum_{E\in\pi}\left(\|(f_0 * g)_E\|_{p_1s_1/(p_1-s_1)}\right)^{s_2}\right]^{1/s_2}\\ &\leq \alpha(m(V))^{1/p_1}\|f_0 * g\|^{\pi}_{(p_1s_1/(p_1-s_1),t_2)} \quad (\text{since by (c) } t_2 \leq s_2)\\ &\leq \alpha(m(V))^{1/p_1}\Gamma(\pi)\|f_0\|^{\pi}_{\underline{r}}\|g\|^{\pi}_{\underline{q}}m(V)^{(1/s_1-1/p_1-1/t_1)} \leq \frac{\varepsilon}{4}\|g\|^{\pi}_{\underline{q}}.\end{aligned}$$

(Again, the result follows similarly for $s_2 = \infty$.) Thus in all cases

$$\|h \cdot (f_0 * g)\|^{\pi}_{\underline{s}} \leq \|(h \cdot (f_0 * g))_{S_\alpha}\|^{\pi}_{\underline{s}} + \|(h(f_0 * g))_{G\sim S_\alpha}\|^{\pi}_{\underline{s}} < \frac{\varepsilon}{2}\|g\|^{\pi}_{\underline{q}}$$

and we have $\|P_hC_{f_0}\|_{\underline{q},\underline{s}} < \varepsilon/2$.

By 5.1, $\|P_hC_{(f-f_0)}\|^{\pi}_{\underline{q},\underline{s}} \leq \Gamma(\pi)\|h\|^{\pi}_{\underline{p}}\|f - f_0\|^{\pi}_{\underline{r}} \leq \theta\Gamma(\pi)\|f - f_0\|^{\pi}_{\underline{r}} < \varepsilon/2$. Finally, $\|P_hC_f\|^{\pi}_{\underline{q},\underline{s}} \leq \|P_hC_{f_0}\|^{\pi}_{\underline{q},\underline{s}} + \|P_hC_{(f-f_0)}\|^{\pi}_{\underline{q},\underline{s}} < \varepsilon$.

Theorem 5.1 tells us that, given the hypotheses on the indices, the bilinear mapping of $L^{\pi}_{\underline{p}} \times P^{\pi}_{\underline{r}}$ to the bounded operators from $L^{\pi}_{\underline{q}}$ to $L^{\pi}_{\underline{s}}$ defined by $(h, f) \to P_hC_f$ is jointly continuous. Theorem 5.3, however, with slightly stronger hypotheses on the indices, yields only that if the ball B_θ of radius θ in $L^{\pi}_{\underline{p}}$ is given the coarser $(1, \infty)$ topology the map from $B_\theta \times L^{\pi}_{\underline{r}}$ to these bounded operators is separately continuous. This is precisely what we need for our results on compactness.

We now show that if P_hC_f is a bounded operator between mixed-norm spaces then h and f must lie in mixed-norm spaces–a result similar to what was proved for C_f in 4.4.

PROPOSITION 5.4. *Suppose P_hC_f is a nontrivial bounded operator from $L^{\pi}_{\underline{q}}$ to $L^{\pi}_{\underline{s}}$, with $\overline{1} \leq \underline{q}, \underline{s} \leq \overline{\infty}$. Then f is in $L^{\pi}_{(1,q_2')}$ and h is in $L^{\pi}_{(s_1,\infty)}$.*

PROOF. By hypothesis $f * g$ exists locally a.e. for all g in $L^{\pi}_{\underline{q}}$, so, by 4.4, f is in $L^{\pi}_{(1,q_2')}$.

Suppose $s_1 < \infty$. Choose g in $K(G)$ such that $f * g \not\equiv 0$. Since $f * g$ is continuous there is a $\delta > 0$ and a relatively compact open set W such that $|(f(g)(x))| \geq \delta$ for x in W. Since $h \cdot (f * g)$ is in $L^{\pi}_{\underline{s}}$, it is locally in L_{s_1}. Thus if $s_1 < \infty$

$$\int_W |h(x)|^{s_1}\,dm(x) \leq \left(\frac{1}{\delta}\right)^{s_1}\int_W |h(x)(f * g)(x)|^{s_1}\,dm(x) < \infty.$$

If $s_1 = \infty$ a similar inequality holds for essential bounds. By translating g we see that h is locally L_{s_1}. If there is a sequence $\{E_n\}$ in π such that $\|h_{E_n}\|_{s_1} \geq n$ we can find $\{x_n\}$ such that $\lim_{n\to\infty}\|h_{Wx_n^{-1}}\|_{s_1} = \infty$ and calculating as above $\|(h \cdot (f * g))_{Wx_n^{-1}}\|_{s_1} \to \infty$ which violate the assumption that P_hC_f is bounded.

Using 5.1 and 5.4 completely settles one case.

COROLLARY 5.5. *P_hC_f is bounded from L_∞ to L_∞ iff h is in L_∞ and f is in L_1.*

Proposition 5.4 leads us to examine the necessity of the hypotheses of 5.1.

We restate the hypotheses:

Let $\overline{1} \leq \underline{p}, \underline{q}, \underline{r}, \underline{s} \leq \overline{\infty}$ be such that

(1) $1/p_1 + 1/r_1 \leq 1 + 1/s_1 - 1/q_1$,

(2) $1/p_2 + 1/r_2 \geqslant 1 + 1/s_2 - 1/q_2$,
(3) $s_1 \leqslant P_1$,
(4) $r_2 \leqslant q_2'$.
Theorem 5.1 says that if h is in $L_{\underline{q}}^{\pi}$ and f is in $L_{\underline{r}}^{\pi}$, $P_h C_f$ is bounded from $L_{\underline{q}}^{\pi}$ to $L_{\underline{s}}^{\pi}$. Proposition 5.4 tells us conversely that if $P_h C_f$ is bounded from $L_{\underline{q}}^{\pi}$ to $L_{\underline{s}}^{\pi}$ then (3) and (4) hold for some choice of $\underline{p}$ and $\underline{r}$ with h in $L_{\underline{p}}^{\pi}$ and f in $L_{\underline{r}}^{\pi}$.

We have found examples of "boundary" cases, of functions h and f which come arbitrarily close to satisfying the conditions of (1) and (2) in which $P_h C_f$ may or may not be bounded.

EXAMPLE 5.6. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences such that $-\frac{1}{2} > \alpha_n > -1$, $\lim_{n\to\infty} \alpha_n = -1$, $\beta_n > 0$ and $\sum_{n=1}^{\infty} \beta_n/(1 + \alpha_n) < \infty$. For $0 < x \leqslant 1$, define $f(x) = \sum_{n=1}^{\infty} \beta_n x^{\alpha_n}$, $h(x) = -\log x$ and let $f(x) = h(x) = 0$ elsewhere. Then $h \in L_p(R)$ for $1 \leqslant p < \infty$ but $h \notin L_\infty(R)$; $f \in L_1(R)$ but $f \notin L_q(R)$ for $1 < q \leqslant \infty$. We examine $P_h C_f$ on $L_2(R)$, i.e. $\underline{q} = \underline{s} = \bar{2}$. Then clearly h and f cannot be members of any pair of spaces $L_{\underline{p}}^{\pi}$, $L_{\underline{r}}^{\pi}$ (respectively) for which the conditions (1) and (2) hold, but for any $\varepsilon > 0$ there are index pairs $\underline{p}$ and $\underline{r}$, such that (1) and (2) are violated by less than ε. By choosing particular g in $L_2(R)$ it is easy to show the sequence $\{\beta_n\}$ can be chosen so that $P_h C_f$ is unbounded. The problem is to show that it can be bounded. For this we need:

LEMMA 5.7. *Let g be a nonnegative function in $L^2(R)$ vanishing on $(0, \infty)$ and let $g\hat{}$ denote the increasing rearrangement of g on $(-\infty, 0)$ (i.e. $g\hat{}(x)$ is the usual decreasing rearrangement evaluated at $-x$). Let f be nonnegative, summable, decreasing on $(0, \infty)$ and vanish on $(-\infty, 0)$. Then $(f * g)_{[0,\infty)} \leqslant (f * g\hat{})_{[0,\infty)}$ and both these functions are decreasing on $(0, \infty)$.*

PROOF. For $x \geqslant 0$, we have
$$(f * g)(x) = \int_0^\infty f(t)g(x - t)\,dt = \int_0^\infty f(x + t)g(-t)\,dt.$$
Since f is decreasing, for $x \geqslant 0$
$$\int_0^\infty f(x + t)g(-t)\,dt \leqslant \int_0^\infty f(x + t)g\hat{}(-t)\,dt.$$
If $0 < x \leqslant y$, then $f(y + t) \geqslant f(x + t) \geqslant 0$ for $t > 0$, so $\int_0^\infty f(y + t)g(-t)\,dt \geqslant \int_0^\infty f(x + t)g(-t)\,dt$ and the functions decrease on $(0, \infty)$.

Suppose $\|g\|_2 = 1$, where g is as in the lemma. Then for $x > 0$
$$1 = \int_0^\infty g^2(-t)\,dt = \int_0^\infty (g\hat{}(-t))^2\,dt \geqslant \int_0^\infty (g\hat{}(-t))^2\,dt \geqslant x \cdot (g\hat{}(-x))^2,$$
so $g\hat{}(-x) \leqslant x^{-1/2}$. Thus, by the lemma, for $x > 0$
$$\begin{aligned}(f * g)(x) &= \int_0^{1-x} \sum \beta_n (x + t)^{\alpha_n} g(-t)\,dt \\ &\leqslant \int_0^\infty \sum \beta_n (x + t)^{\alpha_n} g\hat{}(-t)\,dt \\ &\leqslant \sum \beta_n \int_0^\infty (x + t)^{\alpha_n} t^{-1/2}\,dt \\ &= \sum \beta_n x^{\alpha_n + 1/2} \int_0^\infty (1 + t)^{\alpha_n} t^{-1/2}\,dt.\end{aligned}$$

Write $I_n = \int_0^\infty (1+t)^{\alpha_n} t^{-1/2}\,dt$. Then, again by the lemma,

$$\|h \cdot (f * g)\|_2^2 \leqslant \|h \cdot (f * g\hat{\ })\|_2^2 \leqslant \int_0^1 \left(\sum \beta_n I_n x^{\alpha_n + 1/2} \log x\right)^2 dx$$

$$\leqslant \sum\sum \beta_n \beta_m I_n I_m \int_0^1 \log^2 x\, x^{\alpha_n + \alpha_m + 1}\,dx \leqslant \sum\sum \frac{2\beta_n\beta_m I_n I_m}{(\alpha_n + \alpha_m + 2)^3}.$$

Since $-\frac{1}{2} > \alpha_n > -1$, $\alpha_n\alpha_m < 1$ and $\alpha_n + \alpha_m + 2 > (\alpha_n + 1)(\alpha_m + 1)$. Thus

$$\|h \cdot (f * g)\|_2^2 < 2\left(\sum \frac{\beta_n I_n}{(\alpha_n + 1)^3}\right)^2.$$

Since $\{\beta_n\}$ was only constrained by $\sum \beta_n/(1 + \alpha_n) < \infty$, this last sum can be made bounded by appropriate choice of $\{\beta_n\}$ and $P_h C_f$ is bounded on all positive g in $L^2(R)$ with support in $(-\infty, 0)$.

Suppose the support of g is in $(0, \infty)$ and let $g_x\hat{\ }$ be the decreasing rearrangement on $(0, x)$ of the restriction of g to that interval and again assume $\|g\|_2 = 1$. By previous remarks we know $g_x\hat{\ }(t) \leqslant t^{-1/2}$ and

$$(f * g)(x) \leqslant \int_0^x f(t) g_x\hat{\ }(t)\,dt \leqslant \int_0^x f(t) t^{-1/2}\,dt$$

$$\leqslant \sum \beta_n \int_0^x t^{\alpha_n - 1/2}\,dt \leqslant \sum \beta_n \frac{x^{\alpha_n + 1/2}}{\left(\alpha_n + \frac{1}{2}\right)}.$$

Much as before

$$\|h \cdot (f * g)\|_2^2 \leqslant \sum\sum \frac{\beta_n\beta_m}{\left(\alpha_n + \frac{1}{2}\right)\left(\alpha_m + \frac{1}{2}\right)} \int_0^1 x^{\alpha_n + \alpha_m + 1} \log^2 x\,dx$$

$$\leqslant \sum\sum \frac{2\beta_n\beta_m}{\left(\alpha_n + \frac{1}{2}\right)\left(\alpha_m + \frac{1}{2}\right)(\alpha_n + \alpha_m + 2)^3} \leqslant 2\left(\sum \frac{\beta_n}{\left(\alpha_n + \frac{1}{2}\right)(\alpha_n + 1)^3}\right)^2.$$

Again, $\{\beta_n\}$ can be chosen so this sum is bounded, as well as the previous ones. Since any g in $L^2(R)$ can be constructed as linear combinations of positive functions having supports in the two half-lines, for $\{\beta_n\}$ appropriately chosen we have $P_h C_f$ bounded from L^2 to L^2.

6. Compactness of ***PC*** **operators.** In this section we give necessary and sufficient conditions for compactness of PC operators which satisfy the sufficient conditions for boundedness given in 5.1. We thus consider $P_h C_f$: $L_{\underline{q}}^\pi \to L_{\underline{s}}^\pi$ with h in $L_{\underline{p}}^\pi$ and f in $L_{\underline{r}}^\pi$ where the indices satisfy

(1) $1/p_1 + 1/r_1 \leqslant 1 + 1/s_1 - 1/q_1$,

(2) $1/p_2 + 1/r_2 \geqslant 1 + 1/s_2 - 1/q_2$,

(3) $s_1 \leqslant p_1$,

(4) $r_2 \leqslant q_2'$.

Because we do not know enough about compact sets in L_∞ we sometimes further restrict attention to cases which also satisfy

(5) $s_1 < \infty$ and

(6) $s_2 < \infty$.

If the "global" indices p_2 and r_2 are finite, the operators are always compact. Except for the possibility that one or both of "local" indices p_1 and r_1 may be infinite, the proof is similar to classical results for kernel operators. We include the proof because of this complication.

DEFINITION 6.1. We say a measurable function F on $G \times G$ is in $L^{\pi}_{\underline{u},\underline{v}}(G \times G)$ if $F(x, \cdot)$ is in $L^{\pi}_{\underline{u}}$ for almost all x and the function $x \to \|F(x, \cdot)\|^{\pi}_{\underline{u}}$ is in $L^{\pi}_{\underline{v}}$. The $L^{\pi}_{\underline{v}}$ norm of this latter function will be denoted $\|F\|^{\pi}_{(\underline{u},\underline{v})}$.

LEMMA 6.2. *For $\bar{1} \leq \underline{s}$, $\underline{q} \leq \overline{\infty}$ and F in $L^{\pi}_{\underline{q}',\underline{s}}(G \times G)$, the integral operator T: $L^{\pi}_{\underline{q}} \to L^{\pi}_{\underline{s}}$ defined by $Tg(x) = \int_G F(x, y)g(y)\, dy$ is bounded and $\|T\| \leq \|F\|^{\pi}_{(\underline{q}',\underline{s})}$.*

PROOF. By 3.7(b) and the hypotheses on F, $|Tg(x)| \leq \|F(x, -)\|^{\pi}_{\underline{q}'}\|g\|^{\pi}_{\underline{q}}$ a.e. on G, so $\|Tg\|^{\pi}_{\underline{s}} \leq \|g\|^{\pi}_{\underline{q}}\|F\|^{\pi}_{(\underline{q}',\underline{s})}$.

REMARK 6.3. If F is continuous with compact support, then the transformation defined in the lemma is compact.

PROOF. F can be approximated uniformly by "degenerate" kernels composed from functions in $K(G)$. The norm bound of the lemma then shows that T is approximated in norm by the corresponding operators of finite rank.

THEOREM 6.4. *Under conditions* (1) *through* (5) *above, if $p_2 < \infty$, $r_2 < \infty$ then $P_h C_f$ is compact.*

PROOF. Suppose first that $r_1 < \infty$. By 3.7(e) choose, for any $\varepsilon > 0$, f_ε in $K(G)$ such that $\|f - f_\varepsilon\|^{\pi}_{\underline{r}} \leq \varepsilon/(2\|h\|^{\pi}_{\underline{p}}\Gamma(\pi))$. Choose a finite $\hat{p_1} \leq p_1$ such that $s_1 \leq \hat{p_1}$; $1/\hat{p_1} \leq 1 + 1/s_1 - 1/q_1$. Again by 3.7(e) we can choose h_ε in $K(G)$ such that $\|h - h_\varepsilon\|^{\pi}_{(\hat{p_1},p_2)} < \varepsilon/2\Gamma(\pi)\|f_\varepsilon\|^{\pi}_{(\infty,r_2)}$. Since f_ε is in $L^{\pi}_{(\infty,r_2)}$, $(h - h_\varepsilon)$ is in $L^{\pi}_{(\hat{p_1},p_2)}$, and $\hat{p_1}$ was chosen so that $(\hat{p_1}, p_2)$, $\underline{q}$, (∞, r_2), $\underline{s}$ satisfy the conditions of 5.1, we have

$$\|P_h C_{f_\varepsilon} - P_{h_\varepsilon} C_{f_\varepsilon}\| \leq \Gamma(\pi)\|h - h_\varepsilon\|^{\pi}_{(\hat{p_1},p_2)}\|f\|^{\pi}_{(\infty,r_2)} < \varepsilon/2.$$

But again, using 5.1,

$$\|P_h C_f - P_h C_{f_\varepsilon}\| \leq \Gamma(\pi)\|h\|^{\pi}_{\underline{p}}\|f - f_\varepsilon\|^{\pi}_{\underline{r}} < \varepsilon/2,$$

so $P_h C_f$ can be approximated in norm by the compact $P_{h_\varepsilon} C_{f_\varepsilon}$ and is itself compact.

Now suppose $r_1 = \infty$. If also $p_1 = \infty$, (1) becomes a strict inequality, we can find $\hat{p_1} < \infty$ such that $(\hat{p_1}, p_2)$, $\underline{r}$, $\underline{q}$, $\underline{s}$ satisfy the conditions of the theorem and h is also in $L^{\pi}_{(\hat{p_1},p_2)}$. Thus we may suppose without loss of generality that $p_1 < \infty$. Given $\varepsilon > 0$, we can find h_ε in $K(G)$ such that $\|h - h_\varepsilon\|^{\pi}_{\underline{p}} \leq \varepsilon/2\Gamma(\pi)\|f\|^{\pi}_{\underline{r}}$. Since $p_1 < \infty$ we can find $\hat{r_1} < \infty$ such that $1/\hat{r_1} \leq 1 + 1/s_1 - 1/q_1$. Choose f_ε in $K(G)$ such that $\|f - f_\varepsilon\|^{\pi}_{(\hat{r_1},r_2)} < \varepsilon/(2\Gamma(\pi)\|h_\varepsilon\|^{\pi}_{(\infty,p_2)})$. Since $(f - f_\varepsilon)$ is in $L^{\pi}_{(\hat{r_1},r_2)}$, h_ε is in $L^{\pi}_{(\infty,p_2)}$ and (∞, p_2), $\underline{q}$, $(\hat{r_1}, r_2)$, $\underline{s}$ satisfy the hypotheses of 5.1 we have $\|P_{h_\varepsilon} C_f - P_{h_\varepsilon} C_{f_\varepsilon}\| \leq \Gamma(\pi)\|h_\varepsilon\|^{\pi}_{(\infty,p_2)}\|f - f_\varepsilon\|^{\pi}_{(\hat{r_1},r_2)} \leq \varepsilon/2$. Moreover $\|P_h C_f - P_{h_\varepsilon} C_f\| \leq \Gamma(\pi)\|h - h_\varepsilon\|^{\pi}_{\underline{p}}\|f\|^{\pi}_{\underline{r}} < \varepsilon/2$ and, as before, $P_h C_f$ is compact.

More interesting results are obtained when either p_2 or r_2 is infinite. We now consider these cases when conditions (1) through (6) are met.

LEMMA 6.5. *If $s_1, s_2 < \infty$ and α, β are members of $L_{\underline{s}}^{\pi}$ having disjoint (essential) support, then*

$$\|\alpha\|_{\underline{s}}^{\pi} + \|\beta\|_{\underline{s}}^{\pi} \geqslant \|\alpha \pm \beta\|_{\underline{s}}^{\pi} \geqslant 2^{(1/s_1 + 1/s_2 - 2)}\left[\|\alpha\|_{\underline{s}}^{\pi} + \|\beta\|_{\underline{s}}^{\pi}\right].$$

PROOF. The first inequality is clear. For the second, we know by 2.1

$$\begin{aligned}
\|\alpha \pm \beta\|_{\underline{s}}^{\pi} &= \left[\sum_{E \in \pi}\left(\int_E |\alpha(x) \pm \beta(x)|^{s_1}\, dm(x)\right)^{s_2/s_1}\right]^{1/s_2}\\
&= \left[\sum_{E \in \pi}\left((\|\alpha_E\|_{s_1})^{s_1} + (\|\beta_E\|_{s_1})^{s_1}\right)^{s_2/s_1}\right]^{1/s_2}\\
&\geqslant \left[\sum_{E \in \pi}\left[2^{(1/s_1 - 1)}\|\alpha_E\|_{s_1} + \|\beta_E\|_{s_1}\right]^{s_2}\right]^{1/s_2}\\
&\geqslant 2^{(1/s_1 - 1)}\left[\sum_{E \in \pi}(\|\alpha_E\|_{s_1})^{s_2} + (\|\beta_E\|_{s_1})^{s_2}\right]^{1/s_2}\\
&\geqslant 2^{(1/s_1 + 1/s_2 - 2)}\left[\left(\sum_{E \in \pi}(\|\alpha_E\|_{s_1})^{s_2}\right)^{1/s_2} + \left(\sum_{E \in \pi}(\|\beta_E\|_{s_1})^{s_2}\right)^{1/s_2}\right]\\
&= 2^{(1/s_1 + 1/s_2 - 2)}\left[\|\alpha\|_{\underline{s}}^{\pi} + \|\beta\|_{\underline{s}}^{\pi}\right].
\end{aligned}$$

Since G is an (IN) group, the operator S defined on locally measurable functions by $(Sf)(x) = f^{\sim}(x) = f(x^{-1})$ leaves all the spaces $L_{\underline{p}}^{\pi}(G)$ invariant and is bounded on each. Then $(SC_fSg)(x) = \int_G f(y)g(xy)\, dm(y)$ and $SP_hS = P_{h^{\sim}}$ so $SP_hC_fS = P_{h^{\sim}}SC_fS$. These will be used in the next proposition. We will also need $Q(t, a, f)$ defined for all t and a in G and f in $L_{\underline{r}}^{\pi}$, by a fixed h in $L_{\underline{p}}^{\pi}$ and a fixed g of compact essential support in $L_{\underline{q}}^{\pi}$ by

$$\begin{aligned}
Q(t, a, f) = {}& \|P_hSC_fS({}_ag - {}_tg)\|_{\underline{s}}^{\pi}\\
&- 2^{(1/s_1 + 1/s_2 - 2)}\left[\|P_hSC_fS({}_ag)\|_{\underline{s}}^{\pi} + \|P_hSC_fS({}_tg)\|_{\underline{s}}^{\pi}\right]
\end{aligned}$$

where (1) through (6) all hold, and $r_2 < \infty$.

LEMMA 6.6. $\varliminf_{t \to \infty} Q(t, a, f) \geqslant 0$.

PROOF. Since $r_2 < \infty$, there is a compact set L for any $\delta > 0$ such that $\|f - f_L\|_{\underline{r}}^{\pi} < \delta$. Then

$$\begin{aligned}
&|Q(t, a, f) - Q(t, a, f_L)|\\
&\quad \leqslant \left(\|P_hSC_{(f - f_L)}S({}_ag)\|_{\underline{s}}^{\pi} + \|P_hSC_{(f - f_L)}S({}_tg)\|_{\underline{s}}^{\pi}\right)(1 + 2^{1/s_1 + 1/s_2 - 2})\\
&\quad \leqslant \|P_{h^{\sim}}\|\,\|S\|\,\|S^{-1}\|\left(\|{}_ag\|_{\underline{q}}^{\pi} + \|{}_tg\|_{\underline{q}}^{\pi}\right)(1 + 2^{1/s_1 + 1/s_2 - 2})\|f - f_L\|_{\underline{r}}^{\pi}\\
&\quad \leqslant \|h\|_{\underline{p}}^{\pi}\|S\|^2\lambda\|g\|_{\underline{q}}^{\pi}(1 + 2^{1/s_1 + 1/s_2 - 2})\|f - f_L\|_{\underline{r}}^{\pi}\\
&\quad < \delta\|h\|_{\underline{p}}^{\pi}\|S\|^2\|g\|_{\underline{q}}^{\pi}(1 + 2^{1/s_1 + 1/s_2 - 1}).
\end{aligned}$$

By 3.14, $\lambda > 0$ can be chosen independent of a and t, so by choice of δ this last quantity is less than ε. The essential support of $SC_{f_L}S({}_ag)$ is contained in $a^{-1}KL^{-1}$ and that of $SC_{f_L}S({}_tg)$ in $t^{-1}KL^{-1}$. Take K_ε to be $KL^{-1}LK^{-1}a$; then if t is not in K_ε, $P_hSC_{f_L}S({}_ag)$ and $P_hSC_{f_L}S({}_tg)$ are in $L^\pi_{\underline{s}}$ and have disjoint essential supports. Thus by 6.5, $Q(t, a, f_L) \geqslant 0$. Hence if $t \notin K_2$, $Q(t, a, f) \geqslant -\varepsilon$.

PROPOSITION 6.7. *If* (1) *through* (6) *hold and* $p_2 = \infty$, $f \in L^\pi_{\underline{r}}$, $f \neq 0$, $h \in L^\pi_{\underline{p}}$, *then* P_hC_f *from* $L^\pi_{\underline{q}}$ *to* $L^\pi_{\underline{s}}$ *is not compact if* $h \notin L^\pi_{(s_1,0)}$.

PROOF. By discussion preceding 6.6 and the invariance of $L^\pi_{(s_1,0)}$ under S, it suffices to show P_hSC_fS is not compact.

Choose $g \neq 0$ in $K(G)$ with support L. Then SC_fSg is continuous, so there is an open W and $\varepsilon > 0$ such that $|SC_fSg| \geqslant \varepsilon$ on W. There is a symmetric relatively compact neighborhood U of the identity and x_0 in W such that $x_0U^4 \subset W$. Let π and F be as in 3.4 with $V = U^2$.

If $x_0 \in E' \in \pi$, $x_{E'} \in x_0U^2$, so $E' \subset x_0U^4 \subset W$. Thus

$$\|P_hSC_fSg\|^\pi_{\underline{s}} \geqslant \left[\int_{E'} |h(x)|^{s_1}|(SC_fS)(g)|^{s_1}\, dm(x)\right]^{1/s_1} \geqslant \varepsilon\|h_{E'}\|_{s_1}.$$

If we write $z(E) = x_{E'}x_E^{-1}$, we have similarly

$$\|P_hSC_fS({}_{z(E)}g)\|^\pi_{\underline{s}} \geqslant \left[\int_E |h(x)|^{s_1}|(SC_fS({}_{z(E)}g))(x)|^{s_1}\, dm(x)\right]^{1/s_1} \geqslant \varepsilon\|h_E\|_{s_1}$$

for if $x \in E$, $(SC_fS({}_{z(E)}g))(x) = (SC_fSg)(x_{E'}x_E^{-1}x)$ and $x_{E'}x_E^{-1}x \in x_{E'}U^2 \subset W$. Note that:

(i) If D is a compact set, there is a finite subset, $J(D)$, of π such that if $z(E) \in D$ then $E \in J(D)$.

(ii) Since $h \notin L^\pi_{(s_1,0)}$, there is $\delta > 0$ such that for any finite subset of π, there is a member E of π not in that subset such that $\|h_E\|_{s_1} \geqslant \delta$.

Let $c = \varepsilon\delta 2^{1/s_1+1/s_2-2}$ and choose E_1 in π such that $\|h_{E_1}\|_{s_1} \geqslant \delta$. Let $z_1 = z(E_1)$ and (by 6.6) choose a compact Γ such that $Q(t, z, f) \geqslant -c$ for t outside Γ. By (i) and (ii) we can pick $E_2 \notin J(\Gamma)$ with $\|h_{E_2}\|_{s_1} \geqslant \delta$. Letting $z_2 = z(E_2)$ and using computations made above we see that $\|P_hSC_fS({}_{z_i}g)\|^\pi_{\underline{s}} \geqslant \varepsilon\delta$ for $i = 1, 2$, and $Q(z_2, z_1, f) \geqslant -c$ since z_2 is not in Γ. But by definition of Q we have

$$\|P_hSC_fS({}_{z_1}g) - P_hSC_fS({}_{z_2}g)\|^\pi_{\underline{s}} \geqslant 2^{1/s_1+1/s_2-2}(2\varepsilon\delta) - c = c.$$

Suppose we have chosen $E_1, E_2, \ldots, E_{n-1}$ in π such that, writing z_i for $z(E_i)$ we have, for $i \neq j$, (a) $\|P_hSC_fS({}_{z_i}g)\|^\pi_{\underline{s}} \geqslant \varepsilon\delta$ and (b) $\|P_hSC_f({}_{z_i}g) - P_hSC_fS({}_{z_j}g)\|^\pi_{\underline{s}} \geqslant c$ where i and j run over $1, 2, \ldots, n-1$. By 6.6, choose Γ compact such that $Q(t, z_i, f) \geqslant -c$ for t outside Γ and for $i = 1, 2, \ldots, n-1$. By (i) and (ii), choose $E_n \notin J(\Gamma)$ such that $\|h_{E_n}\|_{s_1} \geqslant \delta$. As above (a) and (b) then hold for $E_1, \ldots, E_n$. We have, by induction, a bounded sequence $\{{}_{z_n}g\}$ in $L^\pi_{\underline{q}}$ such that $\{P_hSC_fS({}_{z_n}g)\}$ has no convergent subsequence, so P_hSC_fS is not compact.

We now prove the main result of this section. We give two equivalent versions.

THEOREM A. *Let $\underline{p}$, $\underline{q}$, $\underline{r}$, $\underline{s}$, satisfy conditions* (1) *through* (6) *and* $p_2 = +\infty$. *For f in $L_{\underline{r}}^{\pi}$, h in $L_{\underline{p}}^{\pi}$.*

(a) *If $s_1 < p_1$, $P_h C_f$ is compact from $L_{\underline{q}}^{\pi}$ to $L_{\underline{s}}^{\pi}$ iff h is in $L_{(1,0)}^{\pi}$.*

(b) *If $s_1 = p_1$, $P_h C_f$ is compact from $L_{\underline{q}}^{\pi}$ to $L_{\underline{s}}^{\pi}$ iff h is in $L_{(p_1,0)}^{\pi}$.*

THEOREM A′. *Under the above conditions, $P_h C_f$ is compact from $L_{\underline{q}}^{\pi}$ to $L_{\underline{s}}^{\pi}$ iff h is in $L_{(s_1,0)}^{\pi}$.*

PROOF OF A. (a) Suppose h is in $L_{(1,0)}^{\pi}$. Then there is a sequence K_n of compact sets such that $\|h - h_n\|_{(1,\infty)}^{\pi} < 1/n$ where h_n is the restriction of h to K_n. The hypotheses of 5.3 hold, so $\|P_h C_f - P_{h_n} C_f\| = \|P_{(h-h_n)} C_f\| \to 0$. On the other hand, $P_{h_n} C_f$ is compact by 6.4 so $P_h C_f$ is also compact.

Conversely, if h is not in $L_{(1,0)}^{\pi}$, it is not in $L_{(s_1,0)}^{\pi}$ so $P_h C_f$ is not compact by 4.7.

(b) If h is in $L_{(p_1,0)}^{\pi}$, we can find h_n as in (a) such that $\|h - h_n\|_{\underline{p}}^{\pi} \to 0$. By 5.1, $\|P_h C_f - P_{h_n} C_f\| \leqslant \Gamma(\pi)\|f\|_{\underline{r}}^{\pi}\|h - h_n\|_{\underline{p}}^{\pi} \to 0$. Since $P_{h_n} C_f$ is compact by 6.4, $P_h C_f$ is also. The converse is 6.7.

PROOF OF A′. The direct part is 6.7. For the converse, let $s_1 = p_1$. By (b) of Theorem A, $P_h C_f$ is compact. For $s_1 < p_1$, h in $L_{(s_1,0)}^{\pi}$ implies h in $L_{(1,0)}^{\pi}$ so, by (a) of Theorem A, $P_h C_f$ is compact.

The case remaining are those in which $r_2 = +\infty$ which implies $q_2 = 1$. We solve the compactness problem for all such cases when G is first countable.

DEFINITION 6.8. Let X be a translation invariant topological vector space of measurable functions. A member of X will be called translation-continuous in X provided its orbits under both left and right translations define continuous maps of G to X.

By 3.13 every member of $L_{\underline{p}}^{\pi}$ is translation-continuous if $\bar{1} \leqslant \underline{p} < \bar{\infty}$. Some members of $L_{\infty}(G)$ are not. We characterize the situation for the case $p_2 = \infty$. (We only sketch the proof.)

LEMMA 6.9. *Let π and F be as in* 3.4. *For f in $L_{(p,\infty)}^{\pi}$, $1 \leqslant p < \infty$, the following are equivalent*:

(a) *f is translation-continuous in $L_{(p,\infty)}^{\pi}$.*

(b) *f is the limit in $L_{(p,\infty)}^{\pi}$ of uniformly continuous functions on G.*

(c) $\{{}_{x_E^{-1}}f_E \colon E \in \pi\}$ *is a compact set in $L_p(V)$.*

PROOF. Clearly uniformly continuous functions are translation-continuous and it follows easily that norm limits in $L_{(p,\infty)}^{\pi}$ also have the property. Let $\{G_\alpha \colon \alpha \in A\}$ be a net of neighborhoods of the identity with $\bigcap_{\alpha \in A} G_\alpha = \{e\}$ and let g_α be the characteristic function of G_α, normalized in L_1. A modification of the classical argument for L_1 shows $g_\alpha * f$ converges to f in $L_{(p,\infty)}^{\pi}$ and clearly $g_\alpha * f$ are uniformly continuous. Thus (a) and (b) are equivalent.

Note that f in $L_{(p,\infty)}^{\pi}$ means precisely that the set $\{{}_{x_E^{-1}}f_E\}$ is bounded in $L_p(V)$ and thus satisfies, as a subset of $L_p(G)$, conditions (a) and (c) of 3.13. It can be shown (we omit the details) that the set satisfies condition (b) of 3.13 iff f is translation-continuous. Thus the set is compact iff f is translation-continuous and (a) is equivalent to (c).

THEOREM B. *Let* $\underline{p}$, $\underline{q}$, $\underline{r}$, $\underline{s}$ *satisfy conditions* (1) *through* (6), $r_2 = +\infty$. *With* f *in* $L^\pi_{\underline{r}}$, $h \neq 0$ *in* $L^\pi_{\underline{p}}$, *for the operator* $P_h C_f$ *to be compact from* $L^\pi_{\underline{q}}$ *to* $L^\pi_{\underline{s}}$ *it is sufficient and, if* G *is first countable, it is necessary that* f *be translation-continuous in the space of bounded local multipliers* $LM^\pi_{q_1,s_1;\infty}$ (*cf. Definition* 3.19).

PROOF. By the hypotheses, we have $q_2 = 1$, $p_2 \leq s_2 < \infty$. Also $1/r_1 + 1/q_1 - 1/s_1 \leq 1 - 1/p_1 \leq 1$. Since f is locally r_1 it follows from Young's inequality that if E and F are relatively compact sets of positive measure, $\Lambda_F C_F$ maps $L_{q_1}(E)$ into $L_{s_1}(F)$ and its norm is bounded by $\|f_{FE^{-1}}\|_{r_1}$. For any H in π,

$$\|f\|^{\pi,H}_{q_1,s_1;\infty} = \sup_{E\in\pi} \|\Lambda_H C_f \Lambda_E\|_{q_1,s_1} \leq \sup_{E\in\pi} \|f_{HE^{-1}}\|_{r_1}$$
$$\leq \sup_{E\in\pi} \sum_{E'\in S(E,H)} \|f_{E'}\|_{r_1} \leq \Gamma(\pi)\|f\|^\pi_{\underline{r}} < \infty$$

(where $S(E, H)$ and $\Gamma(\pi)$ are as in 4.1). Thus f is in $LM^\pi_{q_1,s_1;\infty}$.

Suppose f is translation-continuous. Assume first $p_1 < \infty$ and let g be in the unit ball of $L^\pi_{\underline{q}}$. For any x in G, we have

$$\begin{aligned}\|_x(P_h C_f g) - P_h C_f g\|^\pi_{\underline{s}} &= \|(_x h)\cdot(_x f * g) - h\cdot(f * g)\|^\pi_{\underline{s}}\\ &\leq \|(_x h - h)\cdot(_x f * g)\|^\pi_{\underline{s}} + \|h\cdot[(_x f - f) * g]\|^\pi_{\underline{s}}\\ &\leq \Gamma(\pi)\|_x h - h\|^\pi_{\underline{p}}\|_x f\|^\pi_{\underline{r}} + \|h\|^\pi_{\underline{p}}\|(_x f - f) * g\|^\pi_{\underline{t}}\end{aligned}$$

(as in the proof of 5.1) where $1/t_1 = \max(0, 1/r_1 + 1/q_1 - 1)$ and $1/t_2 = 1/r_2 + 1/q_2 - 1$. By the hypotheses of the theorem, however, we have in this case $t_1 \geq s_1$ and $t_2 = \infty$, so

$$\|_x(P_h C_f g) - P_h C_f g\|^\pi_{\underline{s}} \leq \Gamma(\pi)\|_x h - h\|^\pi_{\underline{p}}\|_x f\|^\pi_{\underline{r}} + \|h\|^\pi_{\underline{p}}\|(_x f - f) * g\|^\pi_{(s_1,\infty)},$$

and we know

$$\begin{aligned}\|(_x f - f) * g\|^\pi_{(s_1,\infty)} &= \sup_{E\in\pi} \sum_{E'\in\pi} \|((_x f - f) * g_{E'})_E\|_{s_1}\\ &\leq \sup_{E\in\pi} \sum_{E'\in\pi} \|\Lambda_E C_{(_x f - f)}\Lambda_{E'}\|_{q_1,s_1}\|g_{E'}\|_{q_1}.\end{aligned}$$

Let N be the compact invariant neighborhood for the (IN) group G. We know that $N \subseteq Uy_1 \cup Uy_2 \cup \dots \cup Uy_k$ and $V \subseteq z_1N \cup z_2N \cup \dots \cup z_rN$ for some finite sets of y's and z's. Then given any E_0 in π any other member E is contained in the set $\bigcup_{i=1}^r \bigcup_{j=1}^k E_0 y_j x_{E_0}^{-1} x_E z_i$ consisting of rk right translates of E_0 (where, as usual, we have π and the set F of x_E as in 3.4). Let $S(E', z) = \{E'' \in \pi: E'z \cap E'' \neq \varnothing\}$. By 3.4, the cardinality of $S(E', z)$ is bounded by $rn_F(N, V)$, since $E'' \in S(E', z) = \bigcup_{i=1}^r (x_{E'} z_i N \cap E'') \neq \varnothing$, and the bound is independent of z and E'. Thus for some choice of kr elements $\{g_j\}$ in G, depending on E, we have

$$\|\Lambda_E C_{({}_x f - f)} \Lambda_{E'}\|_{q_1,s_1} \leqslant \sum_{j=1}^{kr} \|\Lambda_{E_0 g_j} C_{({}_x f - f)} \Lambda_{E'}\|_{q_1,s_1}$$

$$= \sum_{j=1}^{kr} \|\Lambda_{E_0} C_{({}_x f - f)} \Lambda_{E' g_j^{-1}}\|_{q_1,s_1} \leqslant \sum_{j=1}^{kr} \sum_{E'' \in S(E', g_j^{-1})} \|\Lambda_{E_0} C_{({}_x f - f)}\|_{E''}$$

$$\leqslant kr^2 n_F(N, V) \|{}_x f - f\|^{\pi, E_0}_{q_1, s_1; \infty} \quad \text{(by 3.15 and 3.16)}.$$

Using $q_2 = 1$, we have

$$\|({}_x f - f) * g\|^{\pi}_{(s_1,\infty)} \leqslant kr^2 n_F(N, V) \|{}_x f - f\|^{\pi, E_0}_{q_1, s_1; \infty} \Big(\sum_{E' \in \pi} \|g_{E'}\|_{q_1} \Big)$$

$$= kr^2 n_F(N, V) \|{}_x f - f\|^{\pi, E_0}_{q_1, s_1; \infty} \|g\|^{\pi}_{\underline{q}}.$$

This proves that

$$\|{}_x(P_h C_f g) - P_h C_f g\|^{\pi}_{\underline{s}} \leqslant \Gamma(\pi) \|{}_x h - h\|^{\pi}_{\underline{p}} \|{}_x f\|^{\pi}_{\underline{r}} + \|h\|^{\pi}_{\underline{p}} kr^2 n_F(N, V) \|{}_x f - f\|^{\pi, E_0}_{q_1, s_1; \infty}.$$

The first term on the right approaches 0 as x approaches the identity since we are assuming $\underline{p} < \overline{\infty}$, the second term does likewise by hypothesis, and the convergence is uniform for g in the unit ball. Thus the image of the unit ball under $P_h C_f$ satisfies (b) of 3.13 and it clearly satisfies (a) as well.

By (c) of Proposition 3.13, given $\varepsilon > 0$ we can choose a compact set K such that $\|h_{(G \sim K)}\|^{\pi}_{\underline{p}} \leqslant \varepsilon / \Gamma(\pi) \|f\|^{\pi}_{\underline{r}}$. Then $(P_h C_f g)_{G \sim K} = P_{h_{(G \sim K)}} (C_f g)_{G \sim K}$, which has norm bounded by $\|h_{(G \sim K)}\|^{\pi}_{\underline{p}} \Gamma(\pi) \|f\|^{\pi}_{\underline{r}} < \varepsilon$, for all g in the unit ball, as required. Thus the image of the unit ball under $P_h C_f$ is compact if $p_1 < \infty$.

Consider next $p_1 = \infty$, so h is in $L^{\pi}_{(\infty, p_2)}$ with $p_2 < \infty$. For each positive n we can choose Borel B_n such that if h_n is the restriction of h to the complement of B_n then $\|h_n\|^{\pi}_{(\infty, p_2)} < 1/n$. For fixed n we can find h' in $K(G)$ identically 1 on B_n with range in [0, 1]. By the previous case, $P_{h'} C_f$ is compact from $L^{\pi}_{\underline{q}}$ to $L^{\pi}_{\underline{s}}$, so $(P_h - P_{h_n}) C_f = (P_h - P_{h_n}) P_{h'} C_f$ is compact and by 5.1, $\|P_{h_n} C_f\|_{\underline{q}, \underline{s}} \leqslant \Gamma(\pi) \|h_n\|^{\pi}_{(\infty, p_2)} \|f\|^{\pi}_{\underline{r}} \to 0$. Thus $P_h C_f$ is compact which completes the proof of sufficiency.

Before turning to the converse, we note a special case in which the assumption on the indices guarantee the translation-continuity condition, so we can exclude this case in the proof of necessity. Suppose $1/r_1 + 1/q_1 = 1 + 1/s_1$. By hypothesis this implies $p_1 = \infty$ and, since $s_1 < \infty$, $r_1 < \infty$. We will show for any net $\{t_\alpha\} \to e$ we have translation-continuity: $\|{}_{t_\alpha} f - f\|^{\pi, E_0}_{q_1, s_1; \infty} \to 0$. If not, there are nets $\{t_\alpha\} \to e$, $\{E_\alpha\}$ in π, g_α in $L_{q_1}(E_\alpha)$ with $\|g_\alpha\|_{q_1} = 1$ and $\|[({}_{t_\alpha} f - f) * g_\alpha]_{E_0}\|_{s_1} > \delta > 0$. By 4.2(b) we have $\|[({}_{t_\alpha} f - f) * g_\alpha]_{E_0}\|_{s_1} \leqslant C \|({}_{t_\alpha} f - f) * g_\alpha\|^{\pi}_{\underline{s}} \leqslant C\Gamma(\pi) \|{}_{t_\alpha} f - f\|_{\underline{r}} \|g_\alpha\|_{\underline{q}} = C\Gamma(\pi) \|{}_{t_\alpha} f - f\|_{\underline{r}}$ and by 13.5(b) $\|{}_{t_\alpha} f - f\|_{\underline{r}} \to 0$ since $\underline{r} < \overline{\infty}$. This contradicts the supposition that f does not satisfy translation-continuity. Thus we can assume $1/r_1 + 1/q_1 - 1/s_1 < 1$ in proving necessity.

Suppose f is not translation-continuous in $LM^{\pi}_{q_1, s_1; \infty}$. Since G is first countable there is $\delta > 0$ and a sequence $\{t_n\} \to e$ such that $\|{}_{t_n} f - f\|^{\pi, E_0}_{q_1, s_1; \infty} > \delta$. Thus for each n there is E_n in π such that $\|\Lambda_{E_0} C_{({}_t f - f)} \Lambda_{E_n}\|_{q_1, s_1} > \delta$ for $t = t_n$. This means for each n there is g_n in the unit ball of $L_{q_1}(E_n)$ such that $\|[({}_{t_n} f - f) * g_n]_{E_0}\|_{s_1} \geqslant \delta$. Let

$k_n = {}_{t_n}(\Lambda_{E_0} C_f g_n)$, $p_n = \Lambda_{E_0} C_f g_n$. Then

$$k_n - p_n = t_n\left[(\chi_{E_0} - \chi_{t_n E_0}) C_f g_n\right] + \left[({}_{t_n}f - f) * g_n\right]_{E_0}.$$

The above calculations show the norm in $L_{s_1}(E_0)$ of the second term exceeds δ, hence so does its norm in $L_{\underline{s}}^{\pi}$. By continuity of translation (3.13(b)) and the bound of 5.1 for PC operators, the first term is bounded in $L_{\underline{s}}^{\pi}$ norm by a multiple of $\|\chi_{E_0} - \chi_{t_n E_0}\|_\nu \to 0$, where $1/\nu \leqslant 1 + 1/s_1 - 1/r_1 - 1/q_1$. Thus $\|k_n - p_n\|_{\underline{s}}^{\pi} > \delta/2$ infinitely often, the image of the unit ball under the operator $\Lambda_{E_0} C_f$ violates 3.13(b) and hence the operator is not compact. Let A be relatively compact, Borel and of positive measure. If $\Lambda_A C_f$ is compact, so is $\Lambda_{Ax^{-1}} C_f = T_x \Lambda_A T_{x^{-1}} C_f = T_x \Lambda_A C_f T_{x^{-1}}$ (right translation commutes with left convolution). Since $\Lambda_{Ax^{-1}}$ and Λ_{E_0} commute, all $\Lambda_{Ax^{-1}} \Lambda_{E_0} C_f$ are compact and by 3.13

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\|{}_{t_n}(\Lambda_{Ax^{-1}} \Lambda_{E_0} C_f g_n) - \Lambda_{Ax^{-1}} \Lambda_{E_0} C_f g_n\right\|_{\underline{s}}^{\pi} \\ &= \lim_{n\to\infty} \left\|\Lambda_{Ax^{-1}}(k_n - p_n) + (\Lambda_{t_n^{-1}Ax^{-1}} - \Lambda_{Ax^{-1}})\left[{}_{t_n}(\Lambda_{E_0} C_f g_n)\right]\right\|_{\underline{s}}^{\pi}. \end{aligned}$$

As before, by continuity of translation and the bound for PC operators,

$$\left\|(\Lambda_{t_n^{-1}Ax^{-1}} - \Lambda_{Ax^{-1}})\left[{}_{t_n}(\Lambda_{E_0} C_f g_n)\right]\right\|_{\underline{s}}^{\pi} \to 0$$

so we must have $\|\Lambda_{Ax^{-1}}(k_n - p_n)\|_{\underline{s}}^{\pi} \to 0$ for all x. But

$$\begin{aligned} \|\Lambda_{Ax^{-1}}(k_n - p_n)\|_{s_1} &= \int_G \chi_{Ax^{-1}}(y) |k_n(y) - p_n(y)|^{s_1}\, dm(y) \\ &= \int_G \chi_A(y^{-1}) |k_n(y^{-1}x^{-1}) - p_n(y^{-1}x^{-1})|^{s_1}\, dm(y) \\ &= \left[\chi_{A^{-1}} * |k_n - p_n|^{s_1}\right](x^{-1}) \end{aligned}$$

so this last convolution converges pointwise to zero. By Lebesgue's dominated convergence theorem,

$$\begin{aligned} &\int_G \left[\chi_{A^{-1}} * |k_n - p_n|^{s_1}\right](x)\, dm(x) \\ &\qquad = \int_G \int_G \chi_{A^{-1}}(y) |k_n(y^{-1}x) - p_n(y^{-1}x)|^{s_1}\, dm(x)\, dm(y) \\ &\qquad = \int_G \chi_{A^{-1}}(y) \int_G |k_n(y^{-1}x) - p_n(y^{-1}x)|^{s_1}\, dm(y)\, dm(x) \\ &\qquad = m(A^{-1}) \|k_n - p_n\|_{s_1}^{s_1} \end{aligned}$$

converges to zero. Since $k_n - p_n$ has compact support, we have also $\|k_n - p_n\|_{\underline{s}}^{\pi} \to 0$, which contradicts our result that $\|k_n - p_n\|_{\underline{s}}^{\pi} \not\to 0$. Thus $\Lambda_A C_f$ is not compact for any relatively compact Borel A of positive measure. By assumption on h there is such an A on which $|h| > \varepsilon > 0$. There is also a bounded measurable g of compact essential support such that $gh = 1$ on A, so $\Lambda_A C_f = (\Lambda_A P_g) P_h C_f$. If $\Lambda_A C_f$ is not compact, $P_h C_f$ cannot be.

A portion of the above yields

COROLLARY 6.10. *For f in $L^{\pi}_{(r_1,\infty)}$, h in $L^{\pi}_{(\infty,p_2)}$ with $r_1 < \infty$, $p_2 < \infty$, the operator $P_h C_f$ from $L^{\pi}_{(q_1,1)}$ to $L^{\pi}_{(s_1,s_2)}$ is compact if $1/r_1 + 1/q_1 = 1 + 1/s_1 > 1$, and $p_2 \leqslant s_2 < \infty$.*

COROLLARY 6.11. *Under the hypotheses of Theorem* B, *$P_h C_f$ is compact whenever f is a translation-continuous vector in $L^{\pi}_{\underline{r}}$.*

PROOF. The local multiplier norms of Theorem B are dominated by r_1 norms. The conditions of 6.11 is often easier to verify than that of Theorem B. Unfortunately it is not necessary. We give a counterexample using the uniform partition of R into the intervals $E_n = [2n\pi, 2(n+1)\pi)$. Let D_n be the nth Dirichlet kernel:

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2 \sin \frac{1}{2} x}$$

and let d_n be the normalized restriction of $D_{|n|}$ to $L_1(E_n)$. Let $f = \sum_{n=-\infty}^{\infty} d_n$, so f belongs to $L^{\pi}_{(1,\infty)}(R)$.

EXAMPLE 6.12. The function f is translation-continuous in $LM^{\pi}_{2,2;\infty}$ (so, e.g., for any h in $L^{\pi}_{(\infty,1)}$, $P_h C_f$ is compact from $L^{\pi}_{(2,1)}$ to $L^{\pi}_{(2,1)}$ but f is not translation-continuous in $L^{\pi}_{(1,\infty)}$.

PROOF. For g in $L_2(E_k)$, x in E_0,

$$\begin{aligned}(f * g)(x) &= \int_{-\infty}^{\infty} f(y) g(x-y)\, dy = \int_{-\infty}^{\infty} g(y) f(x-y)\, dy \\ &= \int_{2k\pi}^{2(k+1)\pi} f(x-y) g(y)\, dy \\ &= \int_{2k\pi}^{x+2k\pi} d_k(x-y) g(y)\, dy + \int_{x+2k\pi}^{2(k+1)\pi} d_{k+1}(x-y) g(y)\, dy.\end{aligned}$$

For $n \geqslant 0$, $D_{n+1}(x) = D_n(x) + \cos(n+1)x$ and the $L_1(E_k)$ norms are strictly increasing so, writing the reciprocal of this L_1 norm of $D_{|k|}$ as C_k,

$$\begin{aligned}|(f * g)(x)| \leqslant C_k &\left(\left| \int_{2k\pi}^{2(k+1)\pi} D_{|k|}(x-y) g(y)\, dy \right| \right) \\ &+ \left| \int_{x+2k\pi}^{2(k+1)\pi} \cos(|k+1|(x-y)) g(y)\, dy \right|.\end{aligned}$$

As is well known, the first of these integrals is π times the projection of g to a subspace of $L^2(2k\pi, 2(k+1)\pi)$, so

$$\int_0^{2\pi} \left| \int_{2k\pi}^{2(k+1)\pi} D_{|k|}(x-y) g(y)\, dy \right|^2 dx \leqslant \pi^2 \int_{2k\pi}^{2(k+1)\pi} |g(y)|^2\, dy.$$

By direct computation, we also have

$$\int_0^{2\pi} \left| \int_{x+2k\pi}^{2(x+1)\pi} \cos(|k+1|(x-y)) g(y)\, dy \right|^2 dx \leqslant 4\pi^2 \int_{2k\pi}^{2(k+1)\pi} |g(y)|^2\, dy.$$

Thus $\int_0^{2\pi} |(f * g)(x)|^2\, dx \leqslant C_k^2 7\pi^2 \|g\|_2^2$. Since the C_k's converge to zero, f is in $LM^{\pi}_{2,2;0} \subset LM^{\pi}_{2,2;\infty}$. Thus given $\varepsilon > 0$ we can choose a compact set K such that $\sup_x \|{}_x(1 - \Lambda_K) f\|^{\pi, E_0}_{2,2;\infty} < \varepsilon/3$. Since K is compact, as $x \to 0$, $\|{}_x(\Lambda_K f) - \Lambda_K f\|^{\pi, E_0}_{2,2;\infty} \leqslant \|{}_x(\Lambda_K f) - \Lambda_K f\|^{\pi}_{(1,\infty)} \to 0$. Thus if x is small enough $\|{}_x f - f\|^{\pi, E_0}_{2,2;\infty} < \varepsilon$, so f is translation-continuous in $LM^{\pi}_{2,2;\infty}$.

For $0 < \delta < \pi/2$, let $E(\delta) = \bigcup_{n=-\infty}^{\infty}[2n\pi - \delta, 2n\pi + \delta)$ and let $F(\delta)$ be its complement. We know

$$\int_0^{2\pi-\delta} |D_n(x)|\, dx \leqslant \int_\delta^{2\pi-\delta} \frac{dx}{2\sin\left(\frac{1}{2}x\right)} \leqslant \frac{2\pi - 2\delta}{2\sin(\delta/2)},$$

so as $x \to 0$, $\|_x(\Lambda_{F(\delta)}f) - \Lambda_{F(\delta)}f\|^\pi_{(1,\infty)} \to 0$. On the other hand,

$$\|_{2\delta}(\Lambda_{E(\delta)}f) - \Lambda_{E(\delta)}f\|^\pi_{(1,\infty)} = \|_{2\delta}(\Lambda_{E(\delta)}f)\|^\pi_{(1,\infty)} + \|\Lambda_{E(\delta)}f\|^\pi_{(1,\infty)}$$

which can be made arbitrarily close to one. Thus f is not translation-continuous in $L^\pi_{(1,\infty)}$.

7. Summary and applications. In this section we review our results, illustrate important special cases, and show how the results can be extended.

Proposition 3.7 establishes that the mixed-norm spaces $L^\pi_{(p,q)}(G)$ (for $1 \leqslant p, q \leqslant \infty$) provide a class of Banach spaces which interpolate (and include isometrically) the usual $L_p(G)$ spaces as topological vector spaces, have duality relationships similar to those of the L_p spaces, and have the continuous functions of compact support as a dense subset if $p, q < \infty$. Theorem 4.2 provides the analogue of Young's inequality for convolutions on these mixed-norm spaces. (See also **[18]**.) It incidentally establishes that $L^\pi_{(p,1)}(G)$ is always a group algebra and allows us to strengthen a lemma of Stewart **[16]** to show (for G abelian) there is a *bounded* L_1 function on G with Fourier transform of compact support and identically equal to one on a prescribed compact set.

Proposition 5.4 shows the introduction of mixed-norm spaces cannot be avoided, even in studying bounded *PC* operators on L_p, since the product and convolution functions must lie in certain mixed-norm spaces but need not lie in any of the usual L_p spaces. Theorem 5.1 establishes sufficient conditions for boundedness of *PC* operators between mixed-norm spaces. Necessary and sufficient conditions for $P_h C_f$ to be bounded are obtained only for operators from L_∞ to itself (h must be in L_∞ and f in L_1). Example 5.6 shows that if the conditions of 5.1 are approximated arbitrarily closely whether the operator is bounded or not can depend on "rate of growth" conditions.

As an application, consider *PC* operators on $L_2(R)$ (which were the original motivation for this study). Let π be the partition of R into unit intervals $[n, n + 1)$. Proposition 5.4 states that for such an operator $P_h C_f$ to be bounded it is necessary that f be in $L^\pi_{(1,2)}$ and h be in $L^\pi_{(2,\infty)}$, i.e. f must be integrable over each interval and the resulting sequence of $L_1([n, n + 1))$ norms must be square summable while h must be locally square integrable and the sequence of $L_2([n, n + 1))$ norms must be bounded. Theorem 5.1 says such an operator will be bounded if f is in $L^\pi_{(r_1,r_2)}$–locally in $L_{r_1}([n, n + 1))$, with norm sequence in l_{r_2}–and h is in $L^\pi_{(p_1,p_2)}$, where $1/p_1 + 1/r_1 \leqslant 1$, $1/p_2 + 1/r_2 \geqslant 1$, $2 \leqslant p_1$, $r_2 \leqslant 2$. (The last two conditions are also necessary, of course.)

Our discussion of compactness is limited to operators $P_h C_f$ which satisfy the sufficient conditions for boundedness given in Theorem 5.1 acting between spaces where neither index is infinite. In that case Theorem 6.4 shows that all the operators for which the "global" second indices of both h and f are finite are compact. For a PC operator in $L^2(R)$, for instance, the operator will always be compact unless in the above conditions $p_2 = \infty$. Theorem A gives necessary and sufficient conditions for compactness of $P_h C_f$ when the global index of h is infinite. (This, incidentally, answers the question posed in the first paragraph of this paper: if h is in $L_\infty(R)$ and f in $L_1(R)$ then a necessary and sufficient condition for compactness of $P_h C_f$ as an operator in $L_2(R)$ is that h be in $L^{\pi}_{(1,0)}$, i.e., h must be locally in $L_1([n, n + 1))$ and the sequence of L_1 norms must converge to zero. This is equivalent, of course, to the condition given in [2].) The remaining case, when the global index of f is infinite, is dealt with in Theorem B. This can only occur when the global index of the range space is 1, so does not apply to operators on $L_2(R)$. It can be applied to $L_1(R)$, however. For instance, Corollary 6.10 shows that $P_h C_f$ is compact from $L_1(R)$ to $L_1(R)$ provided h is in $L_{(\infty,1)}(R)$ (i.e., h is bounded and the local bounds on $[n, n + 1)$ are summable) and f is in $L_{(1,\infty)}(R)$ (i.e., f is locally integrable and the L_1 norms on $[n, n + 1)$ are bounded).

To produce a particularly nasty example, let θ, ϕ be in L_2 with essential support in [0, 1] and let

$$h(x) = \int_{n=1}^{\infty} \sqrt{n}\, \theta(nx - n^2),$$

$$f(x) = \sum_{n=1}^{\infty} \phi(nx - n^2)/\sqrt{n}\, \log^2(2n).$$

Then $P_h C_f$ is bounded on $L_2(R)$ but not compact. On the other hand, with the same f, if $h(x) = \sum_{n=1}^{\infty} \log n\theta(nx - n^2)$ then $P_h C_f$ is compact. (To be more specific, take $\theta(x) = \phi(x) = 1/\sqrt{x}\, \log(x/2)$ on (0, 1].)

The class of kernels to which our results apply can be broadened by applying some simple manipulations. By taking adjoints, we obtained results for convolution-product operators $C_f P_h$ and consideration of $(P_h C_f)^*(P_h C_f)$ yields results for kernels of the form

$$K(x, t) = \int \bar{f}(x - y)|h(y)|^2 f(y - t)\, dy$$

often encountered in applications. A rather different set of kernels can be obtained, however, by using a change of variables. Let k be a monotone increasing absolutely continuous function on R and define T on $L_2(R)$ by

$$(Tg)(x) = \int_{-\infty}^{\infty} f(k(x) - y)g(y)dy = -(f * g)(k(x)).$$

Then

$$(\|Tg\|_2)^2 = \int_{-\infty}^{\infty} |(f * g)(k(x))|^2\, dx$$

$$= \int_{-\infty}^{\infty} |(f * g)(k)|^2 \left(\frac{dx}{dk}\right) dk = (\|P_h C_f g\|_2)^2$$

where $h = (dx/dk)^{1/2}$. Thus boundedness of T reduces to that of a PC operator. Moreover there are clearly partial isometries U and V such that $P_h C_f = UT$, $T = VP_h C_f$, so T is compact iff $P_h C_f$ is. Similarly, if we let

$$(Sg)(x) = \int_{-\infty}^{\infty} f(x - k(y))g(y)dy$$

we need only observe that S is the adjoint of an operator of the type T defined previously (with f replaced by $\bar{f}^{\sim}$). Its boundedness and compactness, therefore, are also determined by those of a PC operator, namely $P_h C_{\bar{f}^{\sim}}$. Thus our results embrace kernels of the type $f(k(x) - y)$ and $f(x - k(y))$ as well as those of the form $h(x)f(x - y)$ and their adjoints $f(x - y)h(y)$. Obviously, the restrictions to monotone k in the above discussion is overly stringent and can be weakened to demand only that the induced measure be absolutely continuous. The dx/dk in the above calculation then becomes the Radon-Nikodym derivative.

For specific examples, let f be in $L_1(R)$ and define the operators T_1, T_2, S_1 and S_2 on $L_2(R)$ by

$$(T_1 g)(x) = \int_{-\infty}^{\infty} f(e^{|x|} + \sin|x| - y)g(y)\, dy,$$

$$(S_1 g)(x) = \int_{-\infty}^{\infty} f(x - e^{|y|} - \sin|y|)g(y)\, dy,$$

$$(T_2 g)(x) = \int_{-\infty}^{\infty} f(|x| + \log|x| - y)g(y)\, dy,$$

$$(S_2 g)(x) = \int_{-\infty}^{\infty} f(x - |y| - \log|y|)g(y)\, dy.$$

Application of our results as outlined above then shows that T_1 and S_1 are compact while T_2 and S_2 are bounded but not compact.

8. Acknowledgements. The authors are indebted to Professor Jesus Gil de Lamadrid for bringing reference **[18]** to their attention and to Professor Loren Argabright for numerous discussions including the observation that the functions considered in 3.9 provide new equivalent norms.

ADDED IN PROOF. The authors have recently been made aware of several publications by H. G. Feichtinger which relate to mixed norm spaces as discussed in this paper. See, for example, **[19]** and **[20]**.

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